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O I E
P 20.

ORDINARY
DIFFERENTIAL EQUATIONS

AN ELEMENTARY TEXT-BOOK

WITH AN INTRODUCTION TO
LIE'S THEORY OF THE GROUP OF ONE PARAMETER

BY

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PREFACE.

THIS elementary text-book on Ordinary Differential Equations, is an attempt to present as much of the subject as is necessary for the beginner in Differential Equations, or, perhaps, for the student of Technology who will not make a specialty of pure Mathematics. On account of the elementary character of the book, only the simpler portions of the subject have been touched upon at all; and much care has been taken to make all the developments as clear as possible—every important step being illustrated by easy examples.

In one material respect, this book differs from the older text-books upon the subject in the English language: namely, in the methods employed. Ever since the discovery of the Infinitesimal Calculus, the integration of differential equations has been one of the weightiest problems that have attracted the attention of mathematicians. It is not possible to develop a method of integration for *all* differential equations; but it was found possible to give theories of integration for certain classes of these equations; for instance, for the *homogeneous* or for the *linear,* differential equation of the first order. Also, important theories for the linear differential equations of the second or higher orders, have been developed. But all these special theories of integration were regarded by the older mathematicians as *different* theories based upon separate mathematical methods.

Since the year 1870, Lie has shown that it is possible to subordinate all of these older theories of integration to a general *method*: that is, he showed that the older methods were applicable *only* to such differential equations as admit of known infinitesimal transformations. In this way it became possible to derive all of the older theories from a common source: and at the same time, to develop a wider point of view for the general theory of differential equations.

Only a very small part of Lie's extensive and important developments upon these subjects could, however, be presented in a text-book intended for beginners. The memoirs published by Lie on differential equations are to be found in the "Verhandlungen der Gesellschaft der Wissenschaften zu Christiania," 1870-74; in the *Mathematische Annalen*, Vol. II., 24 and 25; and in his *Vorlesungen über Differentialgleichungen mit Belannten Infinitesimalen Transformationen*, edited by Dr. G. Scheffers, Teubner, 1891. Besides these sources of information, the writer had the advantage of hearing, in 1886-87, at the same time with Dr. Scheffers, Prof. Lie's first lectures upon these subjects at the University of Leipzig.

All the methods, depending upon the theory of transformation groups, employed in Chapters III.-V., and IX.-XII. of this book, are due *exclusively* to Prof. Lie.

Lie has also developed elegant theories of integration for Clairaut's and Riccati's equations, as well as for the general linear equation with constant coefficients; but, as an exposition of these theories requires a more extensive preparation than it was considered advisable to give in a purely elementary text-book, the author determined to follow, in the treatment of the above-mentioned equations, the older methods—hoping to present Lie's methods for these equations, as well as some of his more far-reaching theories, in a second volume.

In the preparation of this book the author has made free use of the examples in the current English text-books: and he is under special obligations to the works of Boole, Forsyth, Johnson, and Osborne. The treatment of Riccati's equation, Chapter VII., is substantially that given by Boole.

The arrangement of the matter will be found sufficiently indicated by the table of contents; and an index is given at the end of the book.

The articles in the text printed in small type may be omitted by the reader who is going over the subject for the first time.

JAMES MORRIS PAGE.

Johns Hopkins University,
Baltimore, U.S.A.,
July, 1896.

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CHAPTER I.

THE GENESIS OF THE ORDINARY DIFFERENTIAL EQUATION IN TWO VARIABLES. GEOMETRICAL INTERPRETATION.

1. In the first section of this Chapter, we shall explain what is meant by an ordinary differential equation in two variables, and show how to derive a differential equation from its *complete primitive.*

In the second section, we shall show how ordinary differential equations in two variables may be interpreted geometrically.

SECTION I.

Complete Primitive. Order and Degree of an Ordinary Differential Equation.

2. An equation of the form

$$\omega(x, y)=0 \text{ ....(1)}$$

is ordinarily used to express in algebraic language the fact that one of the two variables x and y is a function of the other. If this equation also contains an arbitrary constant c, its presence is indicated by writing the equation in the form

$$\omega(x, y, c)=0. \text{ ....(1')}$$

By differentiating (1′), we obtain

$$d\omega \equiv \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy = 0\,; \quad\text{..................}(2)$$

and the constant *c may* have been removed by the process of differentiation. If, however, (2) still contains c, it may be eliminated by means of (1′); so that we find, either immediately after the differentiation, or after the elimination, an equation involving x, y, and $\frac{dy}{dx}$, of the general form

$$\mathrm{F}\left(x, y, \frac{dy}{dx}\right) = 0. \quad\text{......................}(3)$$

If we make use, as we shall often do, of the customary abbreviations,

$$y' \equiv \frac{dy}{dx},\ \ y'' \equiv \frac{d^2y}{dx^2},\ \ldots,\ y^{(n)} \equiv \frac{d^ny}{dx^n},$$

the last equation may be written

$$\mathrm{F}(x, y, y') = 0\,; \quad\text{......................}(3)$$

and (3) is called *an ordinary differential equation of the first order in two variables.*

3. If the equation (1) contains *two* independent arbitrary constants, so that it may be written in the form

$$\omega(x, y, c, d) = 0, \quad\text{....................}(1'')$$
$$(c, d, \text{consts.});$$

two successive differentiations of (1″) will give an equation containing y'', from which, by means of (1″) and the equation obtained from (1″) by a first differentiation, both arbitrary constants, c, d, if they are still present, may be eliminated. We obtain thus an equation of the general form

$$\mathrm{F}(x, y, y', y'') = 0, \quad\text{....................}(4)$$

which is called *an ordinary differential equation of the second order in two variables.*

4. The equations (1′) and (1″) from which the differential equations (3) and (4) are obtained, are called the *complete primitives* of (3) and (4), respectively. It is clear that if (1) contained *three* independent arbitrary constants it would give rise to a differential equation of the *third* order; and, in general, we see that the *order* of a differential equation, which is defined as that of the highest derivative in the equation, is the same as the number of independent arbitrary constants in the complete primitive. Thus, if the complete primitive contains n independent arbitrary constants, it will give rise to a differential equation of the n^{th} order.

The *degree* of a differential equation is the same as the degree of the derivative of the highest order in the equation, after the equation has been put into a rational form, and cleared of fractions. Thus the equation

$$(y'^2+1)^3=y''^2$$

is of the *second* order, and of the *second* degree.

From what has been said, it is seen that *to find the differential equation of the n^{th} order corresponding to a primitive containing n arbitrary constants, it is necessary to differentiate the primitive n times successively, and eliminate, between the $n+1$ equations thus obtained, the n arbitrary constants.*

The resulting equation will be the required differential equation of the n^{th} order.

5. The inverse process—usually involving one or more integrations—of finding from a differential equation its complete primitive, is called *solving*, or *integrating*, the differential equation, and the arbitrary constants, which were formerly made to vanish by differentiation and elimination, now reappear as constants of integration. When the equation thus obtained contains exactly n independent arbitrary constants, it is called the *general integral*, or the *complete primitive*, of the differential

equation of the n^{th} order. Thus, if

$$F(x, y, y', \ldots, y^{(n)})=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

be a differential equation of the n^{th} order, its general integral will be an equation of the form

$$\omega(x, y, c_1, \ldots, c_n)=0, \quad\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

where the $c_1, \ldots, c_n$ are independent arbitrary constants. It may be noted that (6) is usually referred to as the *general integral* of (5), when (6) is considered as having been derived from (5); if, however, (5) is considered as having been derived from (6), (6) is referred to as the *complete primitive* of (5).

It is evident from the method of deriving from a complete primitive its corresponding differential equation that the general integral cannot contain more than n independent arbitrary constants; for the general integral would then, being treated as a complete primitive, give rise to a differential equation of an order higher than the n^{th}.

6. If a special numerical value is assigned to each of the arbitrary constants, respectively, of a known general integral of a given differential equation, the resulting equation is called a *particular integral* of the given differential equation. Thus the particular integral is free from all arbitrary constants of integration. For example, if the general integral has the form

$$y-mx-n=0,$$

then the equations

$$y-2x-5=0,$$
$$y-3x-7=0, \text{ etc.},$$

will be particular integrals of the given differential equation.

7. We shall now apply to two simple examples the method of finding the differential equation corresponding to a given complete primitive.

Example 1. It is required to find the differential equation of the first order corresponding to the complete primitive

$$y-cx=0, \quad\text{.................................}(7)$$

where c is an arbitrary constant.

By differentiation, we obtain,

$$dy-cdx=0,$$

or

$$c=\frac{dy}{dx}.$$

Hence, from the first equation,

$$\frac{dy}{dx}=\frac{y}{x}. \quad\text{.................................}(8)$$

This is the differential equation required. If we consider (8) as given, and (7) as having been derived from it—by methods to be explained later—(7) is called the *general integral* of (8). By assigning to c in (7) different numerical values, different *particular integrals* are obtained.

Example 2. It is required to find the differential equation of the second order corresponding to the complete primitive,

$$x^2+2ax+y^2+2by=0. \qquad (a,\ b,\ \text{consts.})$$

By two successive differentiations, we obtain the equations

$$x+a+yy'+by'=0,$$

$$1+y'^2+yy''+by''=0.$$

If a and b are eliminated from these three equations, we find, as the differential equation required,

$$(x^2+y^2)y''-2xy'^3+2yy'^2-2xy'+2y=0.$$

8. It has been shown that to pass from a complete primitive to the corresponding differential equation involves merely the processes of differentiation and elimination; but since the steps of an elimination cannot be retraced, it is a matter of much greater difficulty—if possible at all—to pass from the differential equation to the corresponding complete primitive, or general integral. It will be our object to show how, in a number of the simplest and most important cases, we may, from a given differential equation, deduce its general integral.

SECTION II.

Geometrical Interpretation of Ordinary Differential Equations in Two Variables.

9. If the ordinary differential equation of the first order in x and y,

$$F(x, y, y')=0, \qquad\text{..........................(1)}$$

be written in the solved form,

$$y'=\frac{Y(x, y)}{X(x, y)}, \qquad\text{..........................(2)}$$

where X and Y are supposed to be one-valued functions, it is clear that to any pair of values ascribed to x and y, a fixed value of y' will correspond.

If we consider x and y to be the rectangular coordinates of a point in the plane, y' will represent the numerical value of the tangent of the angle made with the x-axis by the straight line connecting the point (x, y) with the origin of coordinates. Now suppose the point (x, y) to move a short distance in the direction given by y'; in the new position of the point, y' will generally have a new value. Suppose the point to move a short distance in the direction now given by y'; in this third position of (x, y) there will be in general a third value ascribed to y'; the point (x, y) can now be supposed to move a short distance in this last direction—and so on. By this means a figure will be traced of which the limit will be a curve of some kind, when the distances through which the point (x, y) is moved are indefinitely diminished. At every point on this curve the equation

$$y'=\frac{Y}{X} \qquad\text{..........................(2)}$$

is satisfied; that is, if

$$\omega(x, y)=0 \qquad\text{..........................(3)}$$

be the equation to this curve, the equation $\omega=0$ must be a particular integral of equation (2), or of the equivalent equation (1).

The curve traced by a point moving under the above restrictions is therefore called an *integral curve* of the ordinary differential equation (1). If we start from any point not on the curve (3), it is evident that by proceeding as before we get a new integral curve. We might, for instance, take as successive starting points the points on the x-axis—provided that the x-axis does not happen to be itself an integral curve—and it is evident that, in all, ∞^1 different integral curves would be obtained, one passing through every point of ordinary position in the plane. These curves must be represented by an equation of the general form,

$$\omega(x, y, c)=0, \qquad (4)$$

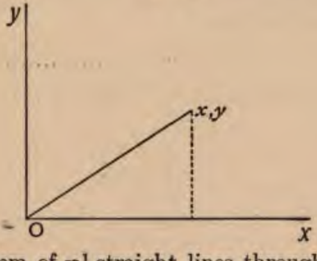
where c is an arbitrary constant, or *parameter*, which assumes different numerical values according as (4) is made to represent the different individual curves of the whole system of integral curves belonging to equation (1). In other words, (4) is the general integral of (1).

Example. The differential equation of the first order

$$x\,dy-y\,dx=0,$$

or,

$$y'=\frac{y}{x} \qquad (5)$$



represents a system of ∞^1 straight lines through the origin. For $\frac{y}{x}$ is the numerical value of the tangent of the angle between the x-axis and the line joining the point (x, y) with the origin; and as y' gives the direction in which the point (x, y) is to be moved, equation (5) asserts that the point (x, y) always moves on the straight line connecting that point with the origin. Since therefore each point of the plane moves on one line of a system of straight lines through the origin, equation (5) represents the family of ∞^1 straight lines

$$\frac{y}{x}=c, \qquad\qquad (6)$$

c being the arbitrary parameter. Thus (6) is the general integral of (5): and the particular integrals are obtained by assigning to c different numerical values.

10. Since the complete primitive, or the general integral, of a differential equation of the second order must contain *two* independent arbitrary constants, or parameters, it is clear that this general integral, or, as we may say, the differential equation of the second order itself, represents geometrically a doubly infinite system of curves in the plane.

Similarly, a differential equation of the third order represents a triply infinite system of curves, etc.

Example. The ordinary differential equation of the second order

$$y''=0 \qquad\qquad (7)$$

asserts that the curvature of the path along which the point (x, y) is to be moved is everywhere zero. Hence the point (x, y) must always describe a straight line, that is, the doubly infinite system of curves which satisfy the above differential equation must be the ∞^2 straight lines of the plane

$$y-mx-n=0. \qquad\qquad (8)$$

It may at once be verified that (8) is the general integral of (7).

EXAMPLES.

Form the differential equations of which the following are the complete primitives, a, b, c being arbitrary constants.

(1) $y=cx$.

(2) $y=cx+\sqrt{1+c^2}$.

(3) $(1+x)^2(1+y)^2=cx^2$.

(4) $y^2-2cx-c^2=0.$

(5) $y=ce^{-\tan^{-1}x}+\tan^{-1}x-1.$

(6) $y=(cx+\log x+1)^{-1}.$

(7) $y=cx+c-c^3.$

(8) $e^{2y}+2cxe^y+c^2=0.$

(9) $y=ax^2+bx.$

(10) $y=c\cos(mx+b).$

(11) $y=x\log\frac{c+bx}{x}.$

(12) $y=cx^3+\frac{b}{x}.$

(13) $y=ae^{2x}+be^{-3x}+ce^x.$

(14) $y=\left(a+bx+\frac{x^2}{2}\right)e^x+c.$

(15) Form the differential equation of the ∞^1 circles having their radii equal to r:

$$(x-a)^2+y^2=r^2.$$

(16) Form the differential equation of *all* circles having their radii equal to r.

(17) Find the differential equation of the family of straight lines which touch the circle

$$x^2+y^2=1;$$

and show that the circle itself also satisfies the differential equation. The equation to the tangents is

$$ax+by-1=0$$

where the constants a and b must satisfy the condition

$$a^2+b^2=1.$$

(18) Find the differential equation of all the conic sections whose axes coincide with the coordinate axes:

$$ax^2+by^2=1.$$

(19) Find the differential equation of all logarithmic spirals around the origin:

$$x^2+y^2=ae^{b\tan^{-1}\frac{x}{y}}.$$

CHAPTER II.

SIMULTANEOUS SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS, AND THE EQUIVALENT LINEAR PARTIAL DIFFERENTIAL EQUATIONS.

11. We shall reserve for a later chapter the consideration of the genesis of an ordinary differential equation in three or more variables, when that equation is obtained from a single primitive by methods similar to those of Chapter I. It will be necessary, however, to give in Secs. I. and II. of this chapter a few propositions relating to simultaneous systems of ordinary differential equations, and the equivalent linear partial differential equations, in order to develop in the next chapter as much of the Theory of Transformation Groups as we shall need.

The third section of this chapter is intended as a supplement to this chapter and to the preceding one. We there indicate, for convenience of reference in Chapter III., the method of integrating the simplest form of an ordinary differential equation in two variables, a problem which really belongs to the Integral Calculus; and we also make a remark upon the integration of the simplest form of a simultaneous system in three variables.

A theory of integration for the general simultaneous system will not be given until Chapter XII.

SECTION I.

The Simultaneous System of Ordinary Differential Equations.

12. Suppose two equations of the form

$$U(x, y, z)=a, \quad V(x, y, z)=b, \quad \ldots\ldots\ldots\ldots(1)$$

are given, where U and V are independent functions of x, y, z, and a and b are arbitrary constants. By differentiating (1) we find

$$\left.\begin{aligned}\frac{\partial U}{\partial x}dx+\frac{\partial U}{\partial y}dy+\frac{\partial U}{\partial z}dz=0,\\ \frac{\partial V}{\partial x}dx+\frac{\partial V}{\partial y}dy+\frac{\partial V}{\partial z}dz=0,\end{aligned}\right\}\ldots\ldots\ldots\ldots(2)$$

as resulting equations.

But from the equations (2) we find that relations of the form

$$\frac{dx}{\frac{\partial U}{\partial y}\frac{\partial V}{\partial z}-\frac{\partial U}{\partial z}\frac{\partial V}{\partial y}}=\frac{dy}{\frac{\partial U}{\partial z}\frac{\partial V}{\partial x}-\frac{\partial U}{\partial x}\frac{\partial V}{\partial z}}=\frac{dz}{\frac{\partial U}{\partial x}\frac{\partial V}{\partial y}-\frac{\partial U}{\partial y}\frac{\partial V}{\partial x}}\ldots(3)$$

must hold: and, if we denote the denominators of these ratios, which are known functions of x, y, z, by $X(x, y, z)$, $Y(x, y, z)$ and $Z(x, y, z)$, respectively, the equations (3) may be written

$$\frac{dx}{X}=\frac{dy}{Y}=\frac{dz}{Z}. \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Thus the system of equations (1), treated as simultaneous complete primitives, gives rise to the so-called *simultaneous system* of ordinary differential equations of the first order, (4).

13. This result in three variables is entirely analogous to that of Art. 2 in two variables. The differential

equation derived in that article from one primitive of the form $U(x, y)=a$ may, of course, be written in a form symmetrical with (4),

$$\frac{dx}{X(x, y)}=\frac{dy}{Y(x, y)}.$$

14. It is obvious that the results of Art. 12 may be extended to n variables.

If

$$U_1(x_1, x_2, \ldots, x_n)=a_1,\ \ U_2(x_1, \ldots, x_n)=a_2,\ \ \ldots,$$
$$U_{n-1}(x_1, \ldots, x_n)=x_{n-1} \ \ldots\ldots(5)$$

be a system of $n-1$ equations in the n variables $x_1, \ldots, x_n$, the $U_1, \ldots, U_{n-1}$ being independent functions of those variables, and the $a_1, \ldots, a_{n-1}$ being arbitrary constants, the system of equations (5), being treated as simultaneous complete primitives, will evidently give rise to a so-called *simultaneous system* of ordinary differential equations of the first order, which may be written in the form

$$\frac{dx_1}{X_1}=\frac{dx_2}{X_2}=\ldots=\frac{dx_n}{X_n}. \ \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Here the $X_1, \ldots, X_n$ are known functions of $x_1, \ldots, x_n$.

In the next section we shall see how the simultaneous system in three variables may be interpreted geometrically.

SECTION II.

Simultaneous Systems and the Equivalent Linear Partial Differential Equations.

15. Equations are of frequent occurrence by means of which a relation between the several *partial* derivatives of a function of two or more variables is expressed. If f be any function of x, y, z, the general form of such

an equation, involving only partial derivatives of f of the first order, and the variables x, y, z, will be

$$F\left(x, y, z, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)=0\,;$$

and if f be known, the values of its partial derivatives substituted in this equation must satisfy the equation identically.

An equation which expresses a relation between the partial derivatives of a function of two or more independent variables—and which may also contain the independent variables themselves explicitly—is called a *partial differential equation*; and the function f, whose partial derivatives satisfy the equation identically, is called the *solution* of the equation.

The *order* and *degree* of a partial differential equation are determined just as are the order and degree of an ordinary differential equation. A partial differential equation of the first order and degree is said to be *linear* of the first order; the term *linear* having reference only to the manner in which the partial derivatives of the solution f enter the equation.

Thus the general form of a linear partial differential equation of the first order in n variables is

$$X_1\frac{\partial f}{\partial x_1}+X_2\frac{\partial f}{\partial x_2}+\ldots+X_n\frac{\partial f}{\partial x_n}=0,$$

where the $X_1, \ldots, X_n$ are certain known functions of the independent variables $x_1, \ldots, x_n$.

We shall hereafter limit ourselves to the consideration of such partial differential equations as are *linear* and of the first order; since this class of equations is, as we shall see, intimately connected with ordinary differential equations.

16. The ordinary differential equation of the first order in two variables may be written in the solved form,

$$\frac{dx}{X(x, y)}=\frac{dy}{Y(x, y)};\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and an intimate relationship may be shown to exist between (1) and the linear partial differential equation in two variables,

$$X(x, y)\frac{\partial f}{\partial x}+Y(x, y)\frac{\partial f}{\partial y}=0. \quad \ldots\ldots\ldots\ldots\ldots(2)$$

For, if $\omega(x, y)=$ const. be the integral of (1), we find by differentiation,

$$\frac{\partial \omega}{\partial x}dx+\frac{\partial \omega}{\partial y}dy=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Now eliminating $\frac{dy}{dx}$ between (3) and (1), we find as a necessary consequence of these equations the identity,

$$X\frac{\partial \omega}{\partial x}+Y\frac{\partial \omega}{\partial y}\equiv 0.$$

That is to say, if the equation $\omega(x, y)=c$ is an integral of the ordinary differential equation (1), ω is also a solution of the linear partial differential equation (2).

Conversely, it may be readily seen that if the function ω is a solution of the linear partial equation (2), $\omega=c$ will also be an integral of (1). Thus the equations (1) and (2) represent fundamentally the same problem, since to find an integral of (1) is the same as to find a solution of (2), and *vice versa*.

17. If the general integral of a given differential equation of the first order (1) has been put into the form

$$\Omega(x, y)=c, \qquad (c=\text{const.})$$

we shall call the function $\Omega(x, y)$ the *integral-function* of the given differential equation.

It is a proposition of the Theory of Functions, which we shall here assume without proof, that an integral-function of a differential equation of the first order always exists, and that all integral-functions of a given differential equation of the first order must be functions of any one of the integral-functions; that is, that no differential equation of the first order, (1), can have two *independent* integral-functions. Thus if U and V be two integral-functions of a given differential equation of the first order, we must be able to express the one as a function of the other, say

$$U=\Phi(V).$$

From this it follows that if we know any integral of a differential equation of the first order, containing an arbitrary constant, we may regard all possible integrals of that equation as known.*

Also, since (1) always has an integral-function, though it cannot have two *independent* integral-functions, the linear partial differential equation of the first order (2) must always have one solution, although it cannot have two independent solutions. The whole number of solutions of (2), or of integral-functions of (1), is evidently unlimited; for if ω be a solution of (2), it is easy to see that any function of ω, as $\Phi(\omega)$, is also a solution of (2).

For, substituting $\Phi(\omega)$ in place of f in (2), we find for that equation

$$\frac{d\Phi}{d\omega}\left(X\frac{\partial\omega}{\partial x}+Y\frac{\partial\omega}{\partial y}\right)=0;$$

but as the expression in parenthesis is zero on account of ω being a solution of (2), the left-hand member of the last equation is zero, that is, $\Phi(\omega)$ is also a solution of (2).

Since every solution of (2) is an integral function of (1), it also follows from this that the most general integral of the ordinary differential equation (1) has the form

$$\Phi(\omega)=\text{const.},$$

where ω is any integral-function of (1).

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* The fact that an ordinary differential equation always has a general integral is illustrated by the types of integrable equations, Chapter IV., as well as by the development, Art. 72, of the general integral in a series.

18. The linear partial differential equation in three variables has the form

$$X(x, y, z)\frac{\partial f}{\partial x}+Y(x, y, z)\frac{\partial f}{\partial y}+Z(x, y, z)\frac{\partial f}{\partial z}=0, \text{......}(4)$$

and it is easy to see that the same relation exists between (4) and a system of equations of the form

$$\frac{dx}{X}=\frac{dy}{Y}=\frac{dz}{Z}, \text{........................}(5)$$

that was seen to exist between (1) and (2).

It is shown in the Theory of Functions that there are always two, and only two *independent* functions of the form $U(x, y, z)$, $V(x, y, z)$, which, when written equal to two arbitrary constants a, b, respectively,

$$U(x, y, z)=a, \quad V(x, y, z)=b, \text{.............}(6)$$

will give, when these equations are differentiated as in Art. 12, values for the ratios dx, dy, dz, which satisfy the simultaneous system (5). When the equations (6) are derived from (5)—by methods to be explained later—they are called the *integrals* of (5).

By differentiation, we find from (6)

$$\frac{\partial U}{\partial x}dx+\frac{\partial U}{\partial y}dy+\frac{\partial U}{\partial z}dz=0,$$

$$\frac{\partial V}{\partial x}dx+\frac{\partial V}{\partial y}dy+\frac{\partial V}{\partial z}dz=0;$$

and these equations, by means of (5) may be written,

$$X\frac{\partial U}{\partial x}+Y\frac{\partial U}{\partial y}+Z\frac{\partial U}{\partial z}=0,$$

$$X\frac{\partial V}{\partial x}+Y\frac{\partial V}{\partial y}+Z\frac{\partial V}{\partial z}=0.$$

But the last two equations show that the functions U

and V, which in accordance with Art. 17 we shall call the *integral-functions* of (5), must be *solutions* of the linear partial differential equation (4). It is thus obvious that any integral-function of (5) must be a solution of (4), and *vice versa*. Hence we see that (4) cannot have more than *two* independent solutions; that is, that every solution of (4) must be capable of being expressed as a function of any two independent solutions of (4).

The whole number of solutions is, however, unlimited; for if U and V are solutions, it is easily seen that any function of U and V, as $\Phi(U, V)$ is also a solution of (4). For, substituting Φ for f in (4) there results

$$\frac{\partial\Phi}{\partial U}\left(X\frac{\partial U}{\partial x}+Y\frac{\partial U}{\partial y}+Z\frac{\partial U}{\partial z}\right)+\frac{\partial\Phi}{\partial V}\left(X\frac{\partial V}{\partial x}+Y\frac{\partial V}{\partial y}+Z\frac{\partial V}{\partial z}\right)=0$$

but since U and V are solutions of (4), the expressions in the parentheses are zero; that is, the last equation is identically satisfied, or $\Phi(U, V)$ is a solution.

Since the solutions of (4) are also integral-functions of (5), the most general integral of (5) has the form

$$\Phi(U, V)=\text{const.},$$

where U and V are any two independent integral-functions.

19. The equations of the preceding article,

$$U(x, y, z)=a, \quad V(x, y, z)=b, \text{ ....}(6)$$

represent two families of ∞^1 surfaces in space; these are the so-called *integral* surfaces of the simultaneous system (5). Also the system of equations (6) may obviously be said to represent a family of ∞^2 curves in space—the curves of intersection of the two families of surfaces—each particular curve being obtained by assigning a pair of special numerical values to the arbitrary constants a and b. One of these curves evidently passes through every general point in space; and at every general point P, on one of these curves, the equations (5) must be satisfied. That is to say, the tangent at the point P

to the curve passing through that point must have a direction of which the direction cosines are proportional to X, Y, Z respectively.

These ∞^2 curves, at every point of which the equations (5) are satisfied, are sometimes designated as the *characteristics* of the linear partial differential equation (4), which is equivalent to the simultaneous system (5).

Example. As a simple example, we may suppose the equations 6) to have the forms

$$x+y+z=a, \quad x^2+y^2+z^2=b^2, \quad \dots\dots\dots\dots (6')$$
$$(a,\ b^2 \text{ consts.})$$

the first equation representing a system of parallel planes, the second a system of concentric spheres around the origin. Thus the simultaneous equations (6′) represent the ∞^2 circles cut from the ∞^1 concentric spheres by the ∞^1 parallel planes.

By the method of Art. 12 we find the simultaneous system to which (6′) give rise by differentiation in the form,

$$dx+dy+dz=0$$
$$xdx+ydy+zdz=0;$$

whence
$$\frac{dx}{z-y}=\frac{dy}{x-z}=\frac{dz}{y-x}.$$

This is of course equivalent to the linear partial differential equation

$$(z-y)\frac{\partial f}{\partial x}+(x-z)\frac{\partial f}{\partial y}+(y-x)\frac{\partial f}{\partial z}=0;$$

and it may be readily verified that the most general solution of this partial differential equation has the form

$$\Phi(x+y+z, \quad x^2+y^2+z^2).$$

20. In a manner entirely analogous to that of Art. 18, it may be seen that the linear partial differential equation of the first order in n variables,

$$X_1\frac{\partial f}{\partial x_1}+X_2\frac{\partial f}{\partial x_2}+\dots+X_n\frac{\partial f}{\partial x_n}=0, \quad \dots\dots\dots (7)$$

where the $X_1, \dots, X_n$ are certain functions of $x_1, \dots, x_n$, represents the same problem as does the simultaneous system of ordinary differential equations

$$\frac{dx_1}{X_1}=\frac{dx_2}{X_2}=\dots=\frac{dx_n}{X_n}. \quad \dots\dots\dots\dots\dots\dots(8)$$

Also it follows from considerations similar to those of Arts. 17 and 18, that (7) cannot have more than $n-1$ independent solutions. If these are of the form

$$U_1(x_1, \dots, x_n), \quad U_2(x_1, \dots, x_n), \quad \dots, \quad U_{n-1}(x_1, \dots, x_n),$$

the U's being put equal to arbitrary constants,

$$U_1=c_1, \ \dots, \ U_{n-1}=c_{n-1},$$

will give the integrals of (8). Moreover the most general solution of (7), or the most general integral function of (8), has the form

$$\Phi(U_1, U_2, \dots, U_{n-1}).$$

SECTION III.

Integration of Ordinary Differential Equations in Two Variables, in which the Variables can be separated by Inspection; and of a Special Form of a Simultaneous System in Three Variables.

21. Although we are not yet ready to present any general theory of integration of ordinary differential equations, it will be necessary for us to call attention here to the fact that when the variables can be separated by inspection in an ordinary differential equation of the first order in two variables, so that the equation may be written

$$\frac{dx}{X(x)}=\frac{dy}{Y(y)}, \quad \dots\dots\dots\dots\dots\dots\dots\dots(1)$$

its complete integration, which is virtually a problem of the Integral Calculus, may be immediately accomplished. The general integral will have the form

$$\int\frac{dx}{X(x)}-\int\frac{dy}{Y(y)}=\text{const.}, \qquad\text{..................}(2)$$

and (2) is considered the general integral of (1), whether the functions in equation (2) can be expressed in a form free from the sign of integration or not.

Of course the differential equations in which the variables may be separated by inspection constitute only a very small class of all ordinary differential equations of the first order in two variables; but we shall see that the integration of these, the simplest possible differential equations of the first order, will, in a future chapter, furnish us with the means of integrating whole classes of very complicated equations.

Example 1. The ordinary equation in two variables

$$(1+x)y\,dx+(1-y)x\,dy=0$$

may be written

$$\frac{1+x}{x}dx+\frac{1-y}{y}dy=0.$$

The general integral will therefore have the form

$$\int\frac{1+x}{x}dx+\int\frac{1-y}{y}dy=\text{const.},$$

which is seen to be

$$\log(xy)+x-y=\text{const.}$$

The given ordinary differential equation is, moreover, equivalent to the linear partial differential equation

$$(1-y)x\frac{\partial f}{\partial x}-(1+x)y\frac{\partial f}{\partial y}=0\,;$$

and it may at once be verified that if

$$\log(xy)+x-y,$$

or any function of this function, be put in place of f in the linear partial differential equation, that equation will be satisfied identically.

Example 2. Given the equation

$$\frac{dx}{\sqrt{1-x^2}}+\frac{dy}{\sqrt{1-y^2}}=0.$$

Here the variables are already separate, and the general integral is,

$$\sin^{-1}x+\sin^{-1}y=a. \qquad (a=\text{const.})$$

But a function of an arbitrary constant is itself an arbitrary constant: hence, taking the sine of both members of the last equation, and replacing $\sin a$ by c, we see that the general integral may be written

$$x\sqrt{1-y^2}+y\sqrt{1-x^2}=c. \qquad (c=\text{const.})$$

It may be readily verified that any function of the integral function

$$x\sqrt{1-y^2}+y\sqrt{1-x^2}$$

is a solution of the linear partial differential equation

$$\sqrt{1-x^2}\frac{\partial f}{\partial x}-\sqrt{1-y^2}\frac{\partial f}{\partial y}=0.$$

22. Similarly, if a given simultaneous system in three variables has the very special form

$$\frac{dx}{X(x)}=\frac{dy}{Y(y)}=\frac{dz}{Z(z)},$$

its integrals may also at once be written in the forms

$$\int\frac{dx}{X(x)}-\int\frac{dy}{Y(y)}=\text{const.},\ \int\frac{dy}{Y(y)}-\int\frac{dz}{Z(z)}=\text{const.}$$

Example. The simultaneous system

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}$$

evidently has for its integrals

$$\log x-\log y=\text{const.},\quad \log y-\log z=\text{const.};$$

or, as they may be written,

$$\frac{x}{y}=a,\ \ \frac{y}{z}=b. \qquad (a,\,b\ \text{const.})$$

It may at once be verified that any function $\Phi\left(\frac{x}{y},\frac{y}{z}\right)$ is a solution of the linear partial differential equation, equivalent to the above simultaneous system,

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=0.$$

23. It may, finally, be noticed that if the given simultaneous system has the particular form

$$\frac{dx}{X(x, y)}=\frac{dy}{Y(x, y)}=\frac{dz}{Z(x, y, z)};$$

and if the integral of the ordinary differential equation in two variables,

$$\frac{dx}{X(x, y)}=\frac{dy}{Y(x, y)},$$

has been found, either by separating the variables, or by methods to be explained later, in the form

$$U(x, y)=c, \qquad (c=\text{const.})$$

then the last equation may be used to eliminate either of the variables x or y, as may be desired for the purpose of integration, from X, Y, or Z. If, for instance, we find from the last equation

$$y=\phi(c, x),$$

the second integral of the given simultaneous system may be found by integrating an ordinary differential equation in two variables of the form

$$\frac{dx}{X(x, \phi)}=\frac{dz}{Z(x, \phi, z)};$$

where, of course, the value of ϕ in terms of x and c has been substituted in place of y.

If the integral of this equation has been found in the form

$$W(x, z, c)=b, \qquad (b=\text{const.})$$

we now substitute for c its value $U(x, y)$, finding the second integral required in the form

$$V(x, y, z)=b.$$

The reader will bear in mind that the above is only a very special form of simultaneous system in three variables. A general theory of integration of such differential equations will be given later; but it is convenient to notice these simplest forms now, in order to make use of them in the next chapter.

Example. Given the simultaneous system

$$\frac{dx}{x^2}=\frac{dy}{xy}=\frac{dz}{z^2}.$$

An integral of
$$\frac{dx}{x^2}=\frac{dy}{xy}$$

is found to be
$$\frac{y}{x}=c. \qquad (c=\text{const.})$$

Hence, in the equation
$$\frac{dy}{xy}=\frac{dz}{z^2}$$

we may put for x, $\frac{y}{c}$. Thus we find

$$\frac{c\,dy}{y^2}=\frac{dz}{z^2},$$

of which the integral is
$$\frac{1}{z}-\frac{c}{y}=b. \qquad (b=\text{const.})$$

Now put for c its value, $\frac{y}{x}$, and we find as the second integral required
$$\frac{1}{z}-\frac{1}{x}=b,$$

or
$$\frac{x-z}{xz}=b.$$

Of course this result might have been obtained directly from

$$\frac{dx}{x^2}=\frac{dz}{z^2},$$

without any intermediate steps.

It may readily be verified that any function of the form

$$\Phi\left(\frac{z}{x}, \frac{x-z}{xz}\right)$$

is a solution of the linear partial differential equation

$$x^2\frac{\partial f}{\partial x}+xy\frac{\partial f}{\partial y}+z^2\frac{\partial f}{\partial z}=0,$$

which is equivalent to the given simultaneous system.

EXAMPLES.

Integrate the following ordinary differential equations of the first order in which the variables may be separated by inspection, giving in each case the equivalent linear partial differential equation in two variables, and verifying that the integral-function of the ordinary equation is a solution of the linear partial equation :

(1) $\dfrac{dy}{dx}=my^2x.$

(2) $\dfrac{dx}{1+x}+\dfrac{dy}{1+y}=0.$

(3) $(y^2+xy^2)dx+(x^2-yx^2)dy=0.$

(4) $\dfrac{x\,dx}{1+y}=\dfrac{y\,dy}{1+x}.$

(5) $\sin x\cos y\,dx=\cos x\sin y\,dy.$

(6) $(1+y^2)dx=(y+\sqrt{1+y^2})(1+x^2)^{\frac{3}{2}}dy.$

(7) $\sec^2x\tan y\,dy+\sec^2y\tan x\,dx=0.$

Give the linear partial differential equations equivalent to the following simultaneous systems ; integrate the simultaneous systems, and show that any function of the integral functions of each simultaneous system is a solution of the corresponding linear partial equation.

(8) $\dfrac{dx}{x}=\dfrac{dy}{y}=\dfrac{dz}{-z}.$

(9) $\dfrac{dx}{-y}=\dfrac{dy}{x}=\dfrac{dz}{z}.$

(10) $\dfrac{dx}{-y}=\dfrac{dy}{x}=\dfrac{dz}{1+z^2}.$

(11) $\dfrac{dx}{yz}=\dfrac{dy}{xz}=\dfrac{dz}{xy}.$

(12) $\dfrac{dx}{0}=\dfrac{dy}{z}=\dfrac{dz}{-y}.$

In (12) the symbol $\dfrac{dx}{0}$ is used merely to show that the coefficient of $\dfrac{\partial f}{\partial x}$ in the linear partial differential equation equivalent to (12) is zero. That partial differential equation is

$$z\frac{\partial f}{\partial y}-y\frac{\partial f}{\partial z}=0\ ;$$

and since $\dfrac{\partial f}{\partial x}$ does not occur at all, it is clear that x is a solution of the equation : that is, $x=$const. is one integral of (12).

CHAPTER III.

THE FUNDAMENTAL THEOREMS OF LIE'S THEORY OF THE GROUP OF ONE PARAMETER.

24. We propose to develop in the present chapter such of the propositions which Lie has established with reference to the transformation group of one parameter, as we shall need subsequently in the integration of ordinary differential equations. The theory of the group of one parameter in two variables is minutely explained in order to enable the reader to make use of the group as an instrument for investigation.

For the sake of greater clearness, we shall generally limit ourselves to two variables in establishing the necessary fundamental propositions; but the method of extending the results to n variables, in such cases as it is desirable, will be sufficiently indicated.

SECTION I.

Finite and Infinitesimal Transformations in the Plane. The Group of one Parameter.

25. By a *transformation of the points of the plane*, we understand *an operation by means of which every point of the plane is conveyed to the position of some point of the same plane.*

The general form of a transformation of the points of the plane is given by the system of equations

$$x_1=\phi(x, y), \quad y_1=\psi(x, y), \qquad\text{.................(1)}$$

where ϕ and ψ are independent functions of x and y. We suppose here that the coordinate axes remain unchanged; but every point of general position (x, y) is conveyed to a new position of which the coordinates are $\phi(x, y)$ and $\psi(x, y)$. If, now, (1) be solved in the form

$$x=\Phi(x_1, y_1), \quad y=\Psi(x_1, y_1), \qquad\text{...............(2)}$$

a transformation is obtained which will evidently carry the point $\phi(x, y)$, $\psi(x, y)$ back to its original position (x, y). The transformation (2) is thus said to be *inverse* to the transformation (1).

The successive performance of the transformations (2) and (1) will evidently give a transformation of the form

$$x_1=x, y_1=y;$$

and the last is called the *identical* transformation. This transformation, therefore, really leaves the position of the point (x, y) unchanged.

26. Suppose that a family of ∞^1 transformations is given by the equations

$$x_1=\phi(x, y, a), \quad y_1=\psi(x, y, a), \qquad\text{.............(3)}$$

where a is a parameter which can assume ∞^1 continuous values. In general, then, it will *not* be the case that the performance of any two transformations of the family (3) successively upon the points of the plane will be equivalent to the performance of a third transformation of the family (3) upon those points. For instance, the equations

$$x_1=a-x, \quad y_1=y$$

represent a family of transformations which do not possess the above peculiarity. For if

$$x_2=a_1-x_1, \quad y_2=y_1$$

be a second transformation of the family, we find, when the two transformations are successively performed upon the point (x, y), that this point assumes a position given by

$$x_2 = a_1 - a + x, \quad y_2 = y.$$

But the transformation given by the last equations does not belong to the original family, of the general form,

$$x_1 = \text{const.} - x, \quad y_1 = y.$$

If, now, $\quad x_1 = \phi(x, y, a), \quad y_1 = \psi(x, y, a)$

be any given transformation of the family (3), and if

$$x_2 = \phi(x_1, y_1, a_1), \quad y_2 = \psi(x_1, y_1, a_1)$$

be a second transformation of that family, then the transformation which results from performing these two successively evidently has the form

$$x_2 = \phi\{\phi(x, y, a),\ \psi(x, y, a), a_1\},$$
$$y_2 = \psi\{\phi(x, y, a),\ \psi(x, y, a), a_1\}.$$

If it happens that the right-hand members of these equations have the general forms

$$\phi\{\phi(x, y, a),\ \psi(x, y, a), a_1\} \equiv \phi\{x, y, \lambda(a, a_1)\},$$
$$\psi\{\phi(x, y, a),\ \psi(x, y, a), a_1\} \equiv \psi\{x, y, \lambda(a, a_1)\},$$

where λ is a constant, or parameter, depending upon a and a_1 alone, then the family of ∞^1 transformations (3) are said to form a *finite continuous group.*

Expressed in words, this condition evidently is that *the result of performing successively any two transformations of the family* (3) *upon the points of the plane must be equivalent to the result of performing a third transformation of that family upon those points.*

We shall add to the above, in the groups which we shall consider, the further condition that to every transformation with the parameter a, of the family (3), we must be able to assign a certain transformation of the family with the parameter a, such that the latter transformation shall be the *inverse* of the former, a being a function of a alone. From this further condition follows immediately that the family (3) must contain the *identical* transformation.

The above can hardly be considered as a new condition, however, for it will be seen to be satisfied *eo ipso* in the groups of which we shall make use; and, in fact, this condition is *always* satisfied *eo ipso* in the groups which are of importance in practical investigations.

Since the family of transformations (3) contains *one* arbitrary parameter, we call it, under the above conditions, *a group of one parameter*; or, symbolically, a G_1.

It is clear that the above definition of a G_1 is independent of the number of variables; that is, if in n variables we have ∞^1 continuous operations given which satisfy the above conditions, these ∞^1 operations are said to form a *group of one parameter*.

Example 1. As an example of a group of one parameter, or a G_1, in two variables, we may take the family of ∞^1 translations along the x-axis given by the equations

$$x_1 = x + a, \quad y_1 = y. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$$(a \equiv \text{parameter.})$$

A second transformation of this family is given by equations of the form

$$x_2 = x_1 + a_1, \quad y_2 = y_1.$$

By eliminating x_1, y_1 from these equations, we find

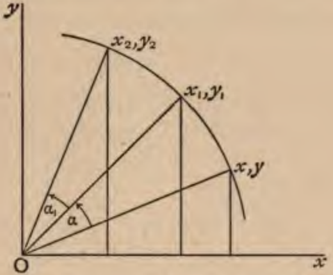
$$x_2 = x + (a + a_1), \quad y_2 = y;$$

and these last equations evidently represent a transformation of the original family (4), namely, the transformation for which the parameter has the value $(a + a_1)$. Hence the family of translations (4) form a G_1: since we see that the effect of performing two of the transformations successively is the same as that of performing a certain third transformation of the family. Also to the transformation with the parameter a we may always assign one with the parameter $-a$, such that the latter is the *inverse* of the former: hence the G_1 contains the identical transformation, which is obviously obtained in this case by assigning to the parameter the value $a - a$, or zero.

Example 2. As a second example of a G_1 we may take the family of ∞^1 rotations of the points of the plane around a fixed point, which we shall assume as the origin of coordinates. The well-known equations to these rotations, as given in Analytical Geometry, are

$$x_1 = x\cos a - y\sin a, \quad\text{..........................(5)}$$
$$y_1 = x\sin a + y\cos a,$$

where a is the angle through which the radii vectors of the points of the plane are turned.



A second transformation of the family (5) is given by the equations

$$x_2 = x_1\cos a_1 - y_1\sin a_1,$$
$$y_2 = x_1\sin a_1 + y_1\cos a_1.$$

By eliminating x_1 and y_1, we find

$$x_2 = (x\cos a - y\sin a)\cos a_1 - (x\sin a + y\cos a)\sin a_1$$
$$= x(\cos a\cos a_1 - \sin a\sin a_1) - y(\sin a\cos a_1 + \cos a\sin a_1);$$

or $$x_2 = x\cos(a + a_1) - y\sin(a + a_1).$$

Similarly, $$y_2 = x\sin(a + a_1) + y\cos(a + a_1).$$

But the last two equations evidently represent a transformation of the family (5), namely the transformation for which the parameter of the family has the value $a + a_1$. Also the *inverse* of the transformation with the parameter a is evidently the transformation with the parameter $-a$. We further find the *identical* transformation in the family by assigning to the parameter the value $a - a$, or zero. Hence, according to the definition, the family of ∞^1 transformations (5) form a *group of one parameter*, or a G_1.

27. Suppose that a G_1 is given by the equations

$$x_1 = \phi(x, y, a), \quad y_1 = \psi(x, y, a); \quad\text{............(6)}$$

we shall show that under the conditions of Art. 26, *the G_1 always contains one infinitesimal transformation.*

For, let a_0 be the value of the parameter for which (6) gives the identical transformation, so that

$$x \equiv \phi(x, y, a_0), \;\; y \equiv \psi(x, y, a_0);$$

If, now, we assign to the parameter a a value which differs from a_0 only by an infinitesimal quantity, say $a_0 + \delta a$, the corresponding transformation

$$x_1 = \phi(x, y, a_0 + \delta a), \;\; y_1 = \psi(x, y, a_0 + \delta a) \text{ ......} (6')$$

will differ only infinitesimally from the identical transformation; that is, (6′) will be an *infinitesimal* transformation.

By Taylor's theorem,

$$x_1 = \phi(x, y, a_0) + \frac{\partial \phi(x, y, a_0)}{\partial a_0}\delta a + \frac{\partial^2 \phi(x, y, a_0)}{\partial {a_0}^2}\frac{\delta a^2}{1 \, . \, 2} + \ldots,$$

$$y_1 = \psi(x, y, a_0) + \frac{\partial \psi(x, y, a_0)}{\partial a_0}\delta a + \frac{\partial^2 \phi(x, y, a_0)}{\partial {a_0}^2}\frac{\delta a^2}{1 \, . \, 2} + \ldots;$$

or, from the above value of the identical transformation,

$$x_1 = x + \frac{\partial \phi(x, y, a_0)}{\partial a_0}\delta a + \frac{\partial^2 \phi(x, y, a_0)}{\partial {a_0}^2}\frac{\delta a^2}{1 \, . \, 2} + \ldots,$$

$$y_1 = y + \frac{\partial \psi(x, y, a_0)}{\partial a_0}\delta a + \frac{\partial^2 \psi(x, y, a_0)}{\partial {a_0}^2}\frac{\delta a^2}{1 \, . \, 2} + \ldots.$$

Thus we see that x_1, y_1 really differ from x and y by infinitesimal quantities.

If the coefficients of all powers of δa up to the r^{th} vanish for all values of x and y in the last equations, we introduce $\delta t \equiv \delta a^r$ as a new infinitesimal quantity, and so obtain the equations of the infinitesimal transformation in the general form

$$x_1 = x + \xi(x, y)\delta t + \ldots, \;\; y_1 = y + \eta(x, y)\delta t + \ldots.$$

Here ξ and η also contain a_0; but since a_0 is a mere number, it is not necessary to write it explicitly in ξ and η.

It is true that by this method for finding the infinitesimal transformation of a given G_1 (6), it is impossible to say whether the succeeding terms of the last equations involve integral or fractional powers of δt; this difficulty is however avoided by a second method given below.

28. Let a fixed value e be assigned to the parameter a in the G_1,

$$x_1=\phi(x, y, a),\ y_1=\psi(x, y, a), \qquad\qquad(6)$$

and suppose that the corresponding transformation, which we shall designate as the transformation (e), carries the point of general position P to the new position P_1. Then, by hypothesis, the transformation in the G_1 (6) which is *inverse* to (e) will carry the point P_1 back to the position P. Now if the parameter of the last transformation be designated by $\bar{e}$, it is clear that a transformation with the parameter $\bar{e}+\delta e$, where δe is an infinitesimal quantity, will carry the point P_1 not exactly back to P, but to a position P' which is at an infinitesimal distance from P. If the transformations (e) and $(\bar{e}+\delta e)$ be performed successively, the result must be equivalent to the performance of a third transformation of the family (6); one that will take the point P *directly* to the position P'. But since the distance PP' is infinitesimal, the transformation which carries the point P directly to the position P' is called an *infinitesimal* transformation.

The above geometrical considerations may be carried out analytically. The first transformation is represented by

$$x_1=\phi(x, y, e),\ y_1=\psi(x, y, e);$$

and the second by

$$x'=\phi(x_1, y_1, \bar{e}+\delta e),\ y'=\psi(x_1, y_1, \bar{e}+\delta e),$$

where we suppose (x, y), (x_1, y_1), and (x', y') to be the coordinates of the three points P, P_1, and P' respectively. The transformation which carries P directly to P' is found by eliminating x_1, y_1 from the above equations: we find

$$x'=\phi\{\phi(x,y,e),\ \psi(x,y,e),\ \bar{e}+\delta e\},\ y'=\psi\{\phi(x,y,e),\ \psi(x,y,e),\ \bar{e}+\delta e\}$$

Developing in powers of δe, we have

$$x'=\phi\{\phi(x, y, e),\ \psi(x,y,e),\ \bar{e}\}+\frac{\partial\phi\{\phi(x,y,e),\ \psi(x,y,e),\ \bar{e}\}}{\partial\bar{e}}\delta e+\dots,$$

$$y'=\psi\{\phi(x, y, e),\ \psi(x,y,e),\ \bar{e}\}+\frac{\partial\psi\{\phi(x,y,e),\ \psi(x,y,e),\ \bar{e}\}}{\partial\bar{e}}\delta e+\dots,$$

But since the transformations (e) and ($\bar{e}$) are inverse, we have the identities,

$$x \equiv \phi\{\phi(x, y, e),\ \psi(x, y, e),\ \bar{e}\},$$
$$y \equiv \psi\{\phi(x, y, e),\ \psi(x, y, e),\ \bar{e}\};$$

and the last two equations become

$$x' = x + \frac{\partial\phi\{\phi(x, y, e),\ \psi(x, y, e),\ e\}}{\partial\bar{e}}\delta e + \ldots,$$

$$y' = y + \frac{\partial\psi\{\phi(x, y, e),\ \psi(x, y, e),\ \bar{e}\}}{\partial\bar{e}}\delta e + \ldots;$$

nd it is evident that these equations represent an *infinitesimal* transformation.

It is easy to see that the coefficients of δe above do not vanish identically; for they may be written

$$\frac{\partial\phi(x_1, y_1, \bar{e})}{\partial\bar{e}},\ \frac{\partial\psi(x_1, y_1, \bar{e})}{\partial\bar{e}}$$

respectively: and if these expressions were identically zero, the equations (6) would necessarily be free of any parameter, which is contrary to hypothesis.

Since $\bar{e}$ depends upon e alone, the equations to the infinitesimal transformation may evidently be written

$$x' = x + \xi(x, y, e)\delta e + \ldots,$$
$$y' = y + \eta(x, y, e)\delta e + \ldots;$$

and it is clear that every G_1 in the plane contains at least *one* infinitesimal transformation.

29. If t be a parameter, it follows from the last two articles that the general form of an infinitesimal transformation in two variables will be

$$\left.\begin{aligned} x_1 &= x + \xi(x, y)\delta t + \ldots \\ y_1 &= y + \eta(x, y)\delta t + \ldots \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

We shall, as usual, neglect higher powers than the first of the infinitesimal quantity δt; and hence the increments which x and y receive by means of the above infinitesimal transformation have the forms

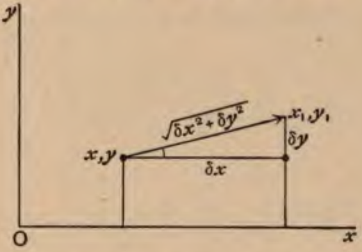
$$\delta x \equiv \xi(x, y)\delta t, \quad \delta y \equiv \eta(x, y)\delta t. \ \ldots\ldots\ldots\ldots\ldots(8)$$

It is clear that this transformation assigns to every point (x, y) of general position, a *direction* through which it is to be moved, given by

$$\frac{\delta y}{\delta x}=\frac{\eta(x, y)}{\xi(x, y)};$$

and also a *distance* through which it is to be moved, given by

$$\sqrt{\delta x^2+\delta y^2}=\sqrt{\xi^2+\eta^2}\,.\,\delta t.$$



As far as determining a direction through which a point of general position is to be moved is concerned, the infinitesimal transformation offers an analogy to the ordinary differential equation of the first order in two variables (Chap. I., Sec. II).

We can get a clear and fruitful idea of an infinitesimal transformation, if we suppose that we put all the points of the plane into motion simultaneously, by performing upon them the infinitesimal transformation (8) an infinite number of times. In this manner a point (x, y) will assume a simply infinite number of continuous positions, which form a curve. The whole change of position of the points of the plane, since it is repeated from moment to moment, may be called a *permanent* motion, and may be compared to the *flow* of the molecules of a compressible fluid.

If t represents the time, and we measure it from a fixed point, say $t=0$ it is clear that the point of general position (x, y) will, after the time t, arrive at a new position (x_1, y_1), where the coordinates x_1, y_1, are functions of x, y, and t. If t increases by dt*, x_1 and y_1 will, by (8), receive the increments

$$dx_1 = \xi(x_1, y_1)dt, \quad dy_1 = \eta(x_1, y_1)dt,$$

so that x_1 and y_1 may be found as functions of t by integrating the simultaneous system

$$\frac{dx_1}{\xi(x_1, y_1)} = \frac{dy_1}{\eta(x_1, y_1)} = dt.$$

The first of these equations has, as we know, an integral of the form

$$U(x_1, y_1) = \text{const.},$$

and by Art. 23, the second equation has for general integral,

$$V(x_1, y_1) - t = \text{const.}$$

Since at the time $t=0$ the point (x_1, y_1) must be at the fixed position (x, y), we must choose the arbitrary constants in the last equations in the forms

$$U(x, y), \quad V(x, y);$$

so that x_1, y_1 are given as functions of t, x, and y, by the equations

$$\left.\begin{aligned} U(x_1, y_1) &= U(x, y), \\ V(x_1, y_1) &= V(x, y) + t, \end{aligned}\right\} \qquad (9)$$

These equations obviously represent a G_1, with the parameter t; and that such must be the case was clear, *a priori*, from the kinematic illustration. For, if in the time t the permanent motion carries the point (x, y) to the position (x_1, y_1)—and in the time t_1 carries the point (x_1, y_1) to the position (x_2, y_2)—it is evident that in the time $t+t_1$ the point (x, y) will be carried to the position

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* We may clearly use either of the symbols δ or d, to indicate an infinitesimal increment. Here we make use of d in order that the simultaneous system may appear in the usual form.

(x_2, y_2); that is to say, the successive performance of any two transformations of the family (9), with the values t and t_1 of the parameter, is equivalent to the performance of a single transformation of the family, with the value $(t+t_1)$ of the parameter. That is, the transformations (9) form a G_1.

Thus we see that every infinitesimal transformation (8) in the plane belongs to a G_1 of finite transformations (9); and the G_1 (9) may be said to be *generated* by the infinitesimal transformation (8), since (9) may be considered as equivalent to the repetition of (8) for an infinite number of times.

It may be shown by means of the Theory of Functions, that every group of one parameter in the plane contains *one* and *only one*, infinitesimal transformation. Thus we may consider the infinitesimal transformation as the *representative* of the G_1.

30. It is clear that a practical method for obtaining the infinitesimal transformation of a given G_1, is to assign to the parameter, in the equations to the finite transformations of the G_1, a value differing only by an infinitesimal quantity from that value which gives the *identical* transformation.

Example 1. The equations to the G_1 of rotations,

$$x_1 = x\cos a - y\sin a,$$
$$y_1 = x\sin a + y\cos a,$$

will obviously represent an identical transformation when the parameter a has the value zero. Thus, if we give to a a value δt, where δt is an infinitesimal quantity, we should obtain the infinitesimal transformation of the above G_1. We find

$$x_1 = x\cos\delta t - y\sin\delta t,$$
$$y_1 = x\sin\delta t + y\cos\delta t;$$

or, by a well-known trigonometrical reduction,

$$x_1 = x - y\,\delta t,$$
$$y_1 = y + x\,\delta t.$$

The last equations, therefore, represent an *infinitesimal rotation.*

Example 2. Suppose the ∞^1 transformations

$$x_1=xt, \quad y_1=yt$$

to be given; it is easy to verify that they form a group of one parameter. Also the identical transformation is obviously given when the parameter t has the value 1. Hence, to find the infinitesimal transformation, we assign to the parameter the value $1+\delta t$; and the infinitesimal transformation of the above G_1 is seen to be

$$x_1=x+x\,\delta t,$$
$$y_1=y+y\,\delta t.$$

31. Since transformations of the nature of those which we have been discussing are certain operations, upon the points of the plane, which are independent of the coordinate axes, it is evident that if equations, representing a G_1, of the form

$$x_1=\phi(x, y, t), \quad y_1=\psi(x, y, t), \quad \ldots\ldots\ldots\ldots(10)$$

are given, these equations must still represent a G_1 when new variables are introduced.

32. We shall now derive a very useful symbol to represent an infinitesimal transformation in the plane.

If $$x_1=\phi(x, y, t), \quad y_1=\psi(x, y, t) \quad \ldots\ldots\ldots\ldots(10)$$

be the finite equations to a G_1, we can evidently consider any function of the form $f(x_1, y_1)$ as a function of x, y, and t; and for that value of t which gives the identical transformation, say for $t=0$, we must have $x_1=x$, $y_1=y$, and hence $f(x_1, y_1)=f(x, y)$. Since $f(x_1, y_1)$ varies when t varies, we are led to inquire as to *what increment*, δf, *the function* $f(x_1, y_1)$ *receives, when* x_1 *and* y_1 *receive their respective increments,*

$$\delta x_1=\xi(x_1, y_1)\delta t, \quad \delta y_1=\eta(x_1, y_1)\delta t.$$

We find

$$\delta f(x_1, y_1)\equiv\frac{\partial f(x_1, y_1)}{\partial x_1}\delta x_1+\frac{\partial f(x_1, y_1)}{\partial y_1}\delta y_1,$$
$$\equiv\left\{\xi(x_1, y_1)\frac{\partial f(x_1, y_1)}{\partial x_1}+\eta(x_1, y_1)\frac{\partial f(x_1, y_1)}{\partial y_1}\right\}\delta t;$$

and this is the increment of the function $f(x_1, y_1)$ under the infinitesimal transformation of the G_1.

If $t=0$, since then $x_1=x$ and $y_1=y$, we must have

$$\delta f(x, y) \equiv \left\{\xi(x, y)\frac{\partial f(x, y)}{\partial x}+\eta(x, y)\frac{\partial f(x, y)}{\partial y}\right\}\delta t.$$

From the form of this increment, it is seen at once that if we know *what increment* a function $f(x, y)$ receives by means of the infinitesimal transformation of a G_1, we may consider the infinitesimal transformation of the G_1 to be *known*, since the increments of x and y under the infinitesimal transformation are precisely the coefficients of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ in the above expression for δf. Hence it is quite natural to introduce the symbol

$$\xi(x, y)\frac{\partial f}{\partial x}+\eta(x, y)\frac{\partial f}{\partial y},$$

to represent the infinitesimal transformation

$$x_1=x+\xi(x, y)\delta t, \quad y_1=y+\eta(x, y)\delta t.$$

Thus, when we speak of the infinitesimal transformation

$$-y\frac{\partial f}{\partial x}+x\frac{\partial f}{\partial y},$$

we mean the infinitesimal rotation

$$x_1=x-y\,\delta t, \quad y_1=y+x\,\delta t.$$

We shall usually represent the above symbol for an infinitesimal transformation still more briefly by the symbol Uf; so that, of course,

$$Uf \equiv \xi(x, y)\frac{\partial f}{\partial x}+\eta(x, y)\frac{\partial f}{\partial y}.$$

Since the infinitesimal transformation of a given G_1 may be considered to *represent* the G_1, Art. 29, we shall often speak simply of the G_1, Uf.

33. It is easy for the reader to convince himself that

an infinitesimal transformation in the variables x, y, z —that is, a transformation by means of which these variables receive the increments

$$\delta x = \xi(x, y, z)\delta t, \quad \delta y = \eta(x, y, z)\delta t, \quad \delta y = \zeta(x, y, z)\delta t,$$

may be represented by the symbol

$$Uf \equiv \xi(x, y, z)\frac{\partial f}{\partial x} + \eta(x, y, z)\frac{\partial f}{\partial y} + \zeta(x, y, z)\frac{\partial f}{\partial z};$$

and the remarks of Arts. 29-32, *mutatis mutandis*, may at once be extended to this transformation in three variables. Thus the above transformation assigns to each point of general position (x, y, z) a distance,

$$\sqrt{\delta x^2 + \delta y^2 + \delta z^2} \equiv \sqrt{\xi^2 + \eta^2 + \zeta^2}\,\delta t$$

through which it is to be moved; and a direction, given by

$$\delta x : \delta y : \delta z = \xi : \eta : \zeta.$$

The symbol of an infinitesimal transformation in n variables is obviously

$$Uf \equiv \xi_1(x_1, \dots x_n)\frac{\partial f}{\partial x_1} + \dots + \xi_n(x_1, \dots x_n)\frac{\partial f}{\partial x_n};$$

and the remarks of Arts. 29-32 as to the geometrical meaning of a transformation, etc., may also be extended to the transformation in n variables.

34. The symbol $U(x)$ means, *put x in place of f in the identity*

$$Uf \equiv \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y};$$

and similarly for $U(y)$. It is seen immediately that

$$U(x) \equiv \xi, \quad U(y) \equiv \eta;$$

so that

$$U(f) \equiv U(x)\,.\,\frac{\partial f}{\partial x} + U(y)\,.\,\frac{\partial f}{\partial y}.$$

35. If new variables x', y' are introduced into the symbol Uf, since

$$\frac{\partial f}{\partial x} \equiv \frac{\partial f}{\partial x'} \cdot \frac{\partial x'}{\partial x} + \frac{\partial f}{\partial y'} \cdot \frac{\partial y'}{\partial x},$$

$$\frac{\partial f}{\partial y} \equiv \frac{\partial f}{\partial x'} \cdot \frac{\partial x'}{\partial y} + \frac{\partial f}{\partial y'} \cdot \frac{\partial y'}{\partial y},$$

Uf becomes

$$\xi\left\{\frac{\partial f}{\partial x'} \cdot \frac{\partial x'}{\partial x} + \frac{\partial f}{\partial y'} \cdot \frac{\partial y'}{\partial x}\right\} + \eta\left\{\frac{\partial f}{\partial x'} \cdot \frac{\partial x'}{\partial y} + \frac{\partial f}{\partial y'} \cdot \frac{\partial y'}{\partial y}\right\},$$

where ξ and η are expressed in terms of x' and y'. But the last expression may evidently be written

$$Uf(x', y') \equiv U(x')\frac{\partial f}{\partial x'} + U(y')\frac{\partial f}{\partial y'}.$$

The method of extending this result to n variables is obvious.

36. We shall now see how convenient the new symbol for an infinitesimal transformation is.

If the function $f(x_1, y_1)$ be developed by Maclaurin's formula, we find, writing f_1 for $f(x_1, y_1)$,

$$f_1 \equiv \left\{f(x_1, y_1)\right\}_{t=0} + \left\{\frac{df_1}{dt}\right\}_{t=0} \cdot \frac{t}{1} + \left\{\frac{d^2 f_1}{dt^2}\right\}_{t=0} \cdot \frac{t^2}{1 \,.\, 2} + \dots \quad (11)$$

Now

$$\frac{df_1}{dt} \equiv \frac{\partial f_1}{\partial x_1} \cdot \frac{dx_1}{dt} + \frac{\partial f_1}{\partial y_1} \cdot \frac{dy_1}{dt};$$

and writing, as we may, the symbol d for the symbol δ to express an infinitesimal increment of x_1 and y_1, we have, Art. 29,

$$\frac{dx_1}{dt} = \xi(x_1, y_1), \quad \frac{dy_1}{dt} = \eta(x_1, y_1).$$

Hence

$$\frac{df_1}{dt} \equiv \xi(x_1, y_1)\frac{\partial f_1}{\partial x_1} + \eta(x_1, y_1)\frac{\partial f_1}{\partial y_1}.$$

But the last expression is exactly the symbol Uf written in the variables x_1, y_1; let us call this U_1f. Thus

$$\frac{df_1}{dt} \equiv U_1f;$$

and this identity means that any function f_1 of x_1 and y_1, gives, when totally differentiated with respect to t, U_1f. But U_1f is itself such a function; hence

$$\frac{d^2f_1}{dt^2} \equiv \frac{d(U_1f)}{dt} \equiv U_1(U_1f);$$

also
$$\frac{d^3f_1}{dt^3} \equiv \frac{d\{U_1(U_1f)\}}{dt} \equiv U_1(U_1(U_1f));$$

and the law of formation of the coefficients in the expansion (11) is now obvious.

If we put $t=0$ in the coefficients of (11), then x_1 and y_1 are changed into x and y; also U_1f becomes Uf; $U_1(U_1f)$ becomes $U(Uf)$, etc. Thus we arrive at the important expansion

$$f(x_1, y_1) = f(x, y) + \frac{t}{1}Uf + \frac{t^2}{1\,.\,2}U(Uf) + \dots \quad \dots\dots(12)$$

This holds, of course, when f_1 has the particular values x_1, and y_1. Thus

$$\left.\begin{aligned} x_1 &= x + \frac{t}{1}U(x) + \frac{t^2}{1\,.\,2}U(U(x)) + \dots \\ y_1 &= y + \frac{t}{1}U(y) + \frac{t^2}{1\,.\,2}U(U(y)) + \dots \end{aligned}\right\} \dots\dots\dots\dots(13)$$

and these are evidently the finite equations of the G_1 of which

$$Uf \equiv \xi(x, y)\frac{\partial f}{\partial x} + \eta(x, y)\frac{\partial f}{\partial y}$$

is the infinitesimal transformation. The equations (13) are of course only another form of the finite equations found, Art. 29, by integrating a simultaneous system. The reader may readily see that the results of this Article may at once be extended, *mutatis mutandis*, to n variables.

Example 1. Suppose the infinitesimal transformation

$$Uf \equiv -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}$$

is given; and we wish to find the finite equations to the G_1.
Here it is seen at once, Art. 34,

$$\begin{array}{rlrl}
U(x) & \equiv -y, & U(y) & \equiv x, \\
U(U(x)) & \equiv -x, & U(U(y)) & \equiv -y, \\
U(U(U(x))) & \equiv y, & U(U(U(y))) & \equiv -x, \\
U(U(U(U(x)))) & \equiv x, & U(U(U(U(y)))) & \equiv y. \\
\cdots\cdots\cdots & & \cdots\cdots\cdots &
\end{array}$$

Thus, by (13),

$$x_1 = x - \frac{t}{1}y - \frac{t^2}{1.2}x + \frac{t^3}{1.2.3}y + \frac{t^4}{1.2.3.4}x - \ldots,$$

$$y_1 = y + \frac{t}{1}x - \frac{t^2}{1.2}y - \frac{t^3}{1.2.3}x + \frac{t^4}{1.2.3.4}y + \ldots;$$

or
$$x_1 = x\left(1 - \frac{t^2}{1.2} + \frac{t^4}{1.2.3.4} - \ldots\right) - y\left(\frac{t}{1} - \frac{t^3}{1.2.3} + \ldots\right),$$

$$y_1 = x\left(\frac{t}{1} - \frac{t^3}{1.2.3} + \ldots\right) + y\left(1 - \frac{t^2}{1.2} + \frac{t^4}{1.2.3.4} - \ldots\right).$$

By well-known developments of the Differential Calculus, the last equations may be written

$$x_1 = x\cos t - y\sin t, \qquad y_1 = x\sin t + y\cos t.$$

Hence the G_1 is the G_1 of rotations, mentioned Art. 26.

Example 2. Given

$$Uf \equiv x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y},$$

to find the finite equations of the G_1.
Here, proceeding as above, we find the expansions

$$x_1 = x + \frac{t}{1}x + \frac{t^2}{1.2}x + \ldots = xe^t,$$

$$y_1 = y + \frac{t}{1}y + \frac{t^2}{1.2}y + \ldots = ye^t.$$

Instead of e^t we may choose a as the parameter of the G_1, and we find as the finite equations,

$$x_1 = ax, \qquad y_1 = ay.$$

In a number of the most important G_1's it will be found that all the terms in the series (13), after the second, are zero.

SECTION II.

Invariance of Functions, Curves, and Equations.

37. Suppose, now, that we demand that a given function of x and y, of the form $\Omega(x, y)$, shall be *invariant* when we perform upon it the transformations of a given G_1. That is, if the infinitesimal transformation of the given G_1 be

$$Uf \equiv \xi(x, y)\frac{\partial f}{\partial x} + \eta(x, y)\frac{\partial f}{\partial y},$$

and the equations to the finite transformations be

$$x_1 = \phi(x, y, t), \quad y_1 = \psi(x, y, t), \quad \text{...........(1)}$$

we demand that when, by means of (1), Ω is expressed as a function of x_1, y_1, Ω must be *the same* function of x_1, y_1 that it was of x, y. Thus we must have, for all values of t,

$$\Omega(x_1, y_1) = \Omega(x, y),$$

by means of (1).

But, from (12) in Sec. I., the last equation may be written

$$\Omega(x, y) + \frac{t}{1}U(\Omega) + \frac{t^2}{1.2}U\big(U(\Omega)\big) + \ldots = \Omega(x, y);$$

and we see that a necessary and sufficient condition that $\Omega(x, y)$ shall be *invariant* under the G_1 (1) is that

$$U(\Omega) \equiv 0. \quad \text{.........................(2)}$$

If this condition be fulfilled, Ω is called an *invariant* of the G_1 (1).

The condition (2) may be written out in full

$$\xi\frac{\partial\Omega}{\partial x}+\eta\frac{\partial\Omega}{\partial y}\equiv 0;$$

and this shows that Ω is a solution of the linear partial differential equation in two variables

$$\xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y}=0,$$

or an integral-function of the equivalent ordinary differential equation

$$\frac{dx}{\xi}=\frac{dy}{\eta}.$$

Hence we see that, by Art. 17, a G_1 in two variables always has one invariant; and every invariant can be expressed as a function of any one invariant.

Example. The function

$$\Omega(x, y)\equiv x^2+y^2$$

is an invariant of the G_1 of rotations;

$$x_1=x\cos t-y\sin t,$$
$$y_1=x\sin t+y\cos t.$$

For, from the last equations,

$$x=x_1\cos t+y_1\sin t,$$
$$y=y_1\cos t-x_1\sin t;$$

hence

$$\begin{aligned}\Omega(x, y)\equiv x^2+y^2&=(x_1\cos t+y_1\sin t)^2+(y_1\cos t-x_1\sin t)^2\\ &=x_1^2(\cos^2 t+\sin^2 t)+y_1^2(\sin^2 t+\cos^2 t)\\ &=x_1^2+y_1^2\equiv\Omega(x_1, y_1).\end{aligned}$$

Hence Ω has the same form in the variables x_1, y_1, for all values of t, that it has in the variables x, y; *i.e.*, Ω is an invariant of the G_1.

The infinitesimal transformation of this G_1 is

$$Uf\equiv -y\frac{\partial f}{\partial x}+x\frac{\partial f}{\partial y};$$

and we may at once verify the fact that $U(\Omega)\equiv 0$; for

$$U(\Omega)\equiv -y\frac{\partial\Omega}{\partial x}+x\frac{\partial\Omega}{\partial y}\equiv -y2x+x2y\equiv 0.$$

We see that the verification of the fact that Ω is an invariant is much simpler when accomplished by means of the infinitesimal transformation of the G_1, than when accomplished by means of the finite transformations.

38. Every point of general position in the plane describes, Art. 29, a continuous curve when the infinitesimal transformation of a given G_1 is performed upon it an infinite number of times. We shall call this curve the *path-curve* of the point under the transformations of the G_1; and it is obvious that each G_1 may be said to have ∞^1 *path-curves*, one through each point of general position in the plane.

The direction through which a point (x, y) is moved by a given G_1, of which the infinitesimal transformation is

$$Uf \equiv \xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y},$$

is given, Art. 29, by

$$\frac{\delta y}{\delta x}=\frac{\eta(x, y)}{\xi(x, y)}.$$

Now if $\Omega(x, y)$ be an invariant of the G_1, we saw that Ω must satisfy the linear partial differential equation

$$U(\Omega) \equiv \xi\frac{\partial \Omega}{\partial x}+\eta\frac{\partial \Omega}{\partial y}=0.$$

But this partial differential equation is equivalent to the ordinary differential equation

$$\xi(x, y)dy-\eta(x, y)dx=0.$$

That is, Ω must be an integral function of the last equation; and the integral curves

$$\Omega(x, y)=\text{const.}$$

have in each point the tangential direction

$$\frac{dy}{dx}=\frac{\eta(x, y)}{\xi(x, y)}.$$

Hence an invariant, $\Omega(x, y)$, of a G_1 in the plane, being written equal to an arbitrary constant, will represent that family of ∞^1 curves in the plane which we call the *path-curves* of the G_1.

39. It should be noticed that any point, or points, in the plane for which

$$\xi(x, y) \equiv \eta(x, y) \equiv 0,$$

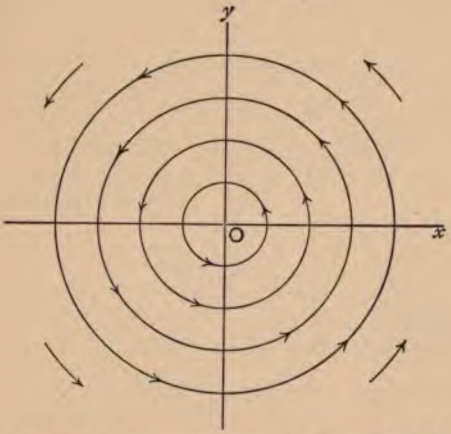
are absolutely invariant under the infinitesimal transformation of the given G_1,

$$Uf \equiv \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y},$$

since in these points x and y do not receive any increments at all.

Example 1. The infinitesimal transformation of the G_1 of rotations is

$$Uf \equiv -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}.$$



Thus, as we know, the invariant must be a solution of the linear partial differential equation

$$-y\frac{\partial \Omega}{\partial x} + x\frac{\partial \Omega}{\partial y} = 0.$$

That is, Ω is the integral-function of

$$\frac{dx}{-y}=\frac{dy}{x},$$

or of

$$xdx+ydy=0.$$

The integral-function of this ordinary differential equation may obviously be assumed to be

$$\Omega\equiv x^2+y^2.$$

Hence the *path-curves* of the G_1, that is, the curves which the points of the plane describe when they are subjected to the transformations of the G_1 of rotations around the origin, are the circles

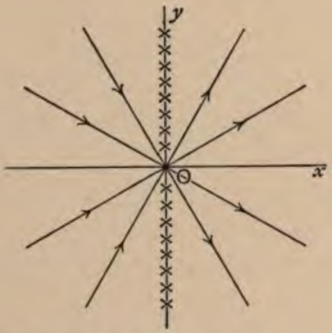
$$\Omega\equiv x^2+y^2=\text{const.}$$

This was, of course, geometrically evident *a priori*. The origin is obviously an absolutely invariant point.

Example 2. Suppose the infinitesimal transformation

$$Uf\equiv x^2\frac{\partial f}{\partial x}+xy\frac{\partial f}{\partial y}$$

to be given.



The invariant is found as the solution of

$$x\frac{\partial\Omega}{\partial x}+y\frac{\partial\Omega}{\partial y}=0,$$

or as the integral-function of

$$xdy-ydx=0.$$

This integral-function is obviously $\Omega\equiv\frac{y}{x}$. Hence the path-curves of the G_1, of which $x^2\frac{\partial f}{\partial x}+xy\frac{\partial f}{\partial y}$ is the infinitesimal transformation, are the straight lines through the origin

$$\frac{y}{x}=\text{const.}$$

The absolutely invariant points are given by

$$x^2=xy=0,$$

that is, $x=0$. Thus the y-axis is an invariant straight line, which consists of absolutely invariant points.

40. A family of ∞^1 curves in the plane, considered as a whole, may be invariant under the transformations of a given G_1 in two ways; each curve of the family may be separately invariant, when, of course, the family is, as a whole, also invariant; or *the curves of the family may, by means of the transformations of the G_1, be interchanged among each other*, leaving the curve-family as a whole, however, still invariant.

We have seen that the path-curves of a given G_1 are a family of ∞^1 curves which is invariant in the first way, that is, each member of the family is separately invariant. Usually, however, when a family of ∞^1 curves in the plane is invariant under the transformations of a given G_1, the individual members of the family are not invariants, but are merely interchanged by means of the transformations of the G_1.

Let

$$\Omega(x, y)=\text{const.}$$

be any family of curves in the plane, which, as a family, are invariant under a G_1 whose finite transformations are given by the equations

$$x_1=\phi(x, y, t),\ \ y_1=\psi(x, y, t),$$

whilst the infinitesimal transformation of the G_1 is

$$Uf\equiv\xi(x, y)\frac{\partial f}{\partial x}+\eta(x, y)\frac{\partial f}{\partial y}.$$

Since the curve-family is to be invariant, the equation to the curves must, in the variables x_1, y_1, have a functional form either identical with, or equivalent to, that in x and y; that is, the equation to the invariant family may be written, in the new variables, in the form

$$\Omega(x_1, y_1)=\text{const.}$$

Now we know that $\Phi(\Omega(x, y))=\text{const.}$ represents the same family of curves that $\Omega(x, y)=\text{const.}$ does; hence we may write, as the condition that the family $\Omega(x,y)=\text{const.}$ shall be invariant,

$$\Omega(x_1, y_1)=\Phi(\Omega(x, y)).$$

If the left-hand member of this equation be developed by means of (12) in Sec. I., we find that a necessary and sufficient condition that the curve-family $\Omega(x, y)=\text{const.}$ shall be invariant, is that

$$U(\Omega(x, y))=\mathrm{F}(\Omega(x, y)).$$

When a relation of this form holds, we sometimes say that the family of curves *admits of* the transformations of the G_1. For the particular case that $\mathrm{F}(\Omega(x, y))\equiv 0$, the above condition gives, as it should, the family of invariant path-curves.

Example 1. We saw that the concentric circles, Art. 39,

$$x^2+y^2=r^2 \qquad (r=\text{const.})$$

are the path-curves of the G_1 of rotations; and hence, of course, they form a family of curves which are invariant under that G_1 in such manner that each curve is separately invariant. But the family of ∞^1 circles is also invariant under the G_1,

$$x_1=xt, \quad y_1=yt,$$

with the infinitesimal transformation

$$Uf\equiv x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}.$$

For, from the above equations,

$$x=\frac{x_1}{t}, \quad y=\frac{y_1}{t};$$

and substituting these values in the equation to the circles, we find

$$\frac{x_1^2+y_1^2}{t^2}=r^2,$$

or

$$x_1^2+y_1^2=\text{const.};$$

which is an equation of the same functional form in x_1, y_1 that the original equation was in x, y.

Thus, by means of the finite transformations of the G_1, we see that the curve-family as a whole is invariant, while the individual members are obviously *not* invariant. We may at once verify the same thing by means of the infinitesimal transformation Uf. For here

$$U(\Omega)\equiv U(x^2+y^2)\equiv 2x\,.\,x+2y\,.\,y=2(x^2+y^2).$$

In this case, therefore,

$$U(\Omega)\equiv 2\Omega,$$

or the curve-family is invariant.

Example 2. The family of straight lines

$$\frac{y}{x}=\text{const.}$$

admit of the G_1 of rotations around the origin. This may be readily verified by means of the finite equations of the rotations. But the infinitesimal transformation is

$$Uf\equiv -y\frac{\partial f}{\partial x}+x\frac{\partial f}{\partial y};$$

and since in this case $\Omega\equiv\frac{y}{x}$, we find

$$U(\Omega)\equiv -y\frac{\partial\Omega}{\partial x}+x\frac{\partial\Omega}{\partial y}$$

$$\equiv\frac{y^2}{x^2}+1\equiv\Omega^2+1.$$

Hence the condition that

$$U(\Omega)\equiv \mathrm{F}(\Omega)$$

holds in this case.

41. The results of Arts. 37-40 may be readily extended, *mutatis mutandis*, to three or more variables.

Thus, if

$$Uf \equiv \xi(x, y, z)\frac{\partial f}{\partial x} + \eta(x, y, z)\frac{\partial f}{\partial y} + \zeta(x, y, z)\frac{\partial f}{\partial z}$$

be the infinitesimal transformation of a G_1 in three variables, the points, or curves, for which

$$\xi \equiv \eta \equiv \zeta \equiv 0,$$

are absolutely invariant under the G_1.

Also, the necessary and sufficient condition that a family of ∞^1 surfaces, $\Omega(x, y, z) = \text{const.}$, shall be invariant under the G_1 is that

$$U(\Omega) \equiv \mathrm{F}(\Omega).$$

42. In a manner entirely analogous to that of Art. 37 it is seen that the necessary and sufficient condition that an *equation* of the form

$$\Omega(x, y) = 0$$

shall be invariant under a given G_1, Uf, is that *the expression* $U(\Omega)$ *shall be zero, either identically or by means of* $\Omega = 0$. This condition may at once be extended to n variables.

Example 1. The equation $\Omega \equiv x^2 + y^2 - 1 = 0$ is invariant under the G_1,

$$Uf \equiv -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}.$$

For here

$$U(\Omega) \equiv -y\frac{\partial \Omega}{\partial x} + x\frac{\partial \Omega}{\partial y} \equiv -2xy + 2xy \equiv 0.$$

Hence the condition for an invariant equation is satisfied.

Example 2. The equation

$$\Omega \equiv y - x = 0$$

is invariant under

$$Uf \equiv x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}.$$

For here

$$U(\Omega) \equiv x\frac{\partial \Omega}{\partial x} + y\frac{\partial \Omega}{\partial y} \equiv -x + y \equiv \Omega.$$

Hence the condition is satisfied.

43. We shall now *find all equations of the general form* $\Omega=0$, *which are invariant under, or "admit of," a given* G_1, Uf; and as this result is very important for future use in more than three variables, we shall develop it at once in n variables.

If the given G_1, in the n variables $x_1, \ldots, x_n$, have the form

$$Uf \equiv \xi_1(x_1, \ldots, x_n)\frac{\partial f}{\partial x_1} + \ldots + \xi_n(x_1, \ldots, x_n)\frac{\partial f}{\partial x_n},$$

it might be possible that the $\xi_1, \ldots, \xi_n$, are such function that they all become zero by means of an equation which is invariant under the G_1. If we represent the equation by

$$\Omega(x_1, \ldots, x_n)=0,$$

it is true that in this case $\Omega=0$ is an invariant equation; but the system of values of the variables which satisfy $\Omega=0$ is not transformed at all.

For instance, in two variables, the equation

$$x^2+y^2-1=0$$

is evidently invariant under the G_1,

$$Uf \equiv (x^2+y^2-1)\frac{\partial f}{\partial x}+2(x^2+y^2-1)\frac{\partial f}{\partial y},$$

inasmuch as the infinitesimal transformation of the G_1 vanishes entirely when x^2+y^2-1 is zero; and the G_1 does not transform at all the system of values of x and y, which satisfy the equation

$$x^2+y^2-1=0.$$

We shall, in future, exclude from consideration an invariant equation which makes all the $\xi_1, \ldots, \xi_n$ identically zero.

Thus we may assume that one at least of the $\xi_1, \ldots, \xi_n$ in Uf does not become zero by means of the equation $\Omega=0$. Let us assume that ξ_n is not zero; then, by

Art. 42, it is clear that if $\Omega=0$ is invariant under the infinitesimal transformation Uf, it will also be invariant under the transformation

$$Yf\equiv\frac{\xi_1}{\xi_n}\frac{\partial f}{\partial x_1}+\frac{\xi_2}{\xi_n}\frac{\partial f}{\partial x_2}+\ldots+\frac{\xi_{n-1}}{\xi_n}\frac{\partial f}{\partial x_{n-1}}+\frac{\partial f}{\partial x_n}.$$

For, if $U(\Omega)$ is zero, either identically, or by means of $\Omega=0$, it is clear that $Y(\Omega)$, which is $U(\Omega)$ divided by ξ_n, will also be zero, either identically or by means of $\Omega=0$.

Now the linear partial differential equation of the first order in n variables,

$$Yf=0,$$

has $(n-1)$ independent solutions which are functions of $x_1, \ldots, x_n$, and which we shall designate as

$$y_1,\ y_2,\ \ldots,\ y_{n-1}.$$

But if we consider x_n in connection with these $(n-1)$ independent functions, it is clear that the n functions

$$y_1,\ y_2,\ \ldots,\ y_{n-1},\ x_n$$

must also be independent. Otherwise we might express x_n as a function of $y_1, \ldots, y_{n-1}$, say in the form

$$x_n\equiv W(y_1,\ \ldots,\ y_{n-1}).$$

But, Art. 20, the last equation means that x_n must be a solution of the linear partial equation $Yf=0$; which is manifestly impossible, since for $f\equiv x_n$ this equation reduces to $1\equiv 0$.

Hence the n functions $y_1, \ldots, y_{n-1}, x_n$ are independent, and we may introduce them as n new independent variables. By Art. 35, it will be easily seen that Yf then assumes the form

$$\frac{\partial f}{\partial x_n},$$

which is a mere translation.

Hence, we may remark, incidentally, that by a proper choice of variables, *every* infinitesimal transformation may be brought to the form of a mere translation.

In the new variables the equation $\Omega=0$ has the form

$$\mathrm{F}(y_1, \ldots, y_{n-1}, x_n)=0,$$

and x_n can only occur *formally* in this equation. For if x_n be really present, we might solve and find x_n in terms of $y_1, \ldots, y_{n-1}$, so that the invariant equation will have the form

$$\mathrm{F} \equiv x_n-\Phi(y_1, \ldots, y_{n-1})=0.$$

But for this equation to be invariant under Yf, we must have $Y(\mathrm{F})$ *zero*, either identically or by means of $\mathrm{F}=0$. Now

$$Y(\mathrm{F}) \equiv Y(x^n-\Phi) \equiv 1\,;$$

and hence we see that the variable x_n cannot occur in the function F.

If now we return to our original variables and designate the equation which is invariant under Uf by $\Omega=0$, it is clear that Ω *must be capable of being expressed as a function of the* $(n-1)$ *independent solutions*

$$y_1, y_2, \cdots, y_{n-1}$$

of the linear partial differential equation $Yf=0$, *or of its equivalent equation* $Uf=0$.

This is a result of much importance for our subsequent investigations.

For the special case of three variables, it follows that to find the most general equation which is invariant under a given G_1,

$$Uf \equiv \xi(x, y, z)\frac{\partial f}{\partial x}+\eta(x, y, z)\frac{\partial f}{\partial y}+\zeta(x, y, z)\frac{\partial f}{\partial z},$$

it will be necessary to find two independent solutions of the linear partial differential equation of the first order

$$Uf \equiv \xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y}+\zeta\frac{\partial f}{\partial z}=0.$$

If these solutions be $u(x, y, z)$ and $v(x, y, z)$, the most general invariant equation will have the form

$$F(u, v)=0;$$

or, written in a form solved for u,

$$u=f(v).$$

SECTION III.

The Lineal Element. The Extended Group of One Parameter.

44. A *lineal element* is the aggregate of a point (x, y) in the plane, and a direction through that point. If y' represents the tangent of the angle which the direction makes with the x-axis, it is clear that x, y, y' may be regarded as the coordinates of the lineal element; and by assigning to y', which need not necessarily be considered a differential coefficient, all possible numerical values, we evidently obtain the ∞^1 lineal elements which pass through the point (x, y).

An ordinary *differential* equation of the first order in two variables, of the form

$$\Omega(x, y, y')=0,$$

may now be considered as an *algebraic* equation in the three variables x, y, y', defining ∞^2 of the ∞^3 lineal elements of the plane. The equation $\Omega=0$, as a differential equation, has ∞^1 integral curves; and the tangent to an integral curve at any point (x, y) must be determined by a value of y' which satisfies the above equation. But the *same* value of y' determines the lineal element through the point (x, y); for when x and y are fixed, only that value of y' will satisfy $\Omega=0$. Thus the ∞^2 lineal elements which are defined by the algebraic equation in three variables, $\Omega=0$, *envelope* the integral curves of the differential equation in two variables, $\Omega=0$, as indicated in Fig. 1.

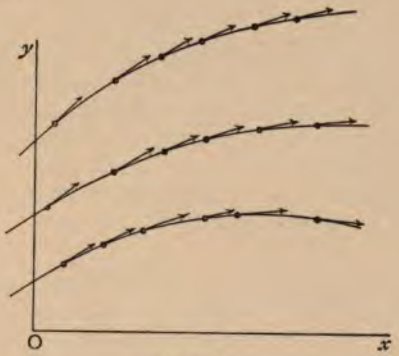


FIG. 1.

If the equation $\Omega=0$ happens not to contain y' at all, it still represents ∞^2 lineal elements, although it can no longer be considered a differential equation. These are evidently the ∞^2 lineal elements whose points lie along the curve $\Omega=0$, as indicated in Fig. 2. Through each point pass ∞^1 lineal elements, since at that point x and y are fixed, while y', being indeterminate, may have ∞^1 different values.

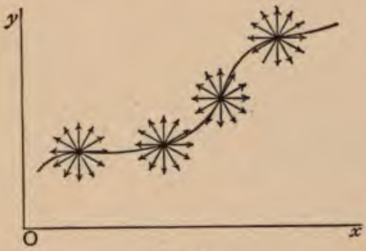


FIG. 2.

In the following, as we have only to do with differential equations, we shall always consider that Ω actually contains y'.

45. If a transformation be given by the equations

$$x_1 = \phi(x, y), \quad y_1 = \psi(x, y), \quad \dots\dots\dots\dots(1)$$

it is obvious that not only the points of the plane, but also the ∞^3 lineal elements are transformed by (1) according to a fixed law. For the value of the transformed y', which we shall call y'_1, and which determines the direction of the transformed lineal element, is determined by means of the equations,

$$y'_1 \equiv \frac{dy_1}{dx_1} \equiv \frac{d\psi}{d\phi} \equiv \frac{\dfrac{\partial\psi}{\partial x} + \dfrac{\partial\psi}{\partial y} \cdot y'}{\dfrac{\partial\phi}{\partial x} + \dfrac{\partial\phi}{\partial y} \cdot y'}.$$

Thus it is seen that the value of y'_1 depends merely upon the transformation (1) and the values assigned to x, y, and y'. The transformation in the three variables x, y, and y',

$$x_1 = \phi(x, y), \quad y_1 = \psi(x, y), \quad y'_1 = \frac{d\psi}{d\phi}, \quad \dots\dots\dots(2)$$

which tells how the ∞^3 lineal elements of the plane are transformed, is called the *extended* transformation corresponding to the transformation (1).

46. If a G_1 be given by the equations

$$x_1 = \phi(x, y, a), \quad y_1 = \psi(x, y, a), \quad \dots\dots\dots\dots(3)$$

each of its ∞^1 transformations may be extended in the above manner. It is natural to expect that the ∞^1 transformations in the variables x, y, y' will themselves form a G_1; and the proof that such is the case is very simple. For let

$$x_2 = \phi(x_1, y_1, a_1), \quad y_2 = \psi(x_1, y_1, a_1) \quad \dots\dots\dots(4)$$

be a second transformation of the G_1; and let the elimination of x_1 and y_1 between (3) and (4) give a transformation of the G_1 of the form

$$x_2=\phi(x, y, b),\ \ y_2=\psi(x, y, b),\ \ldots\ldots\ldots\ldots(5)$$

b being a function of a and a_1 alone.

If each of the transformations (3), (4), and (5) be extended, it is easy to see that all the extended transformations form a G_1. For (3), when extended, becomes

$$x_1=\phi(x, y, a),\ \ y_1=\psi(x, y, a),\ \ y'_1=\frac{d\psi(x, y, a)}{d\phi(x, y, a)};\ \ldots(6)$$

and (4) becomes

$$x_2=\phi(x_1, y_1, a_1),\ \ y_2=\psi(x_1, y_1, a_1),\ \ y'_2=\frac{d\psi(x_1, y_1, a_1)}{d\phi(x_1, y_1, a_1)}.\ (7)$$

The successive performance of (6) and (7) upon the lineal elements of the plane is equivalent to the performance upon them of the transformation obtained by eliminating x_1, y_1 between (6) and (7). But by (5), the latter transformation must have the form

$$x_2=\phi(x, y, b),\ \ y_2=\psi(x, y, b),\ \ y'_2=\frac{d\psi(x, y, b)}{d\phi(x, y, b)},\ \ldots .(8)$$

where b is a function of a and a_1 alone. It is clear that (8) is the transformation which would be obtained by extending (5); that is, the ∞^1 extended transformations, corresponding to the G_1 (3), form themselves a G_1.

47. It is also obvious that if a point transformation of the form (1) be given, not only will y', but also $y'', \ldots, y^{(n)}$, be transformed by (1) according to fixed laws.

The transformation in four variables,

$$x_1=\phi(x, y),\ \ y_1=\psi(x, y),\ \ y'_1=\frac{d\psi}{d\phi},\ \ y''_1=\frac{d^2\psi}{d\phi^2},$$

is called the *twice-extended* transformation corresponding to (1). Each of the ∞^1 transformations of the G_1 (1) may be *twice extended* in this manner; and it is very easy to see that the ∞^1 twice-extended transformations in the four variables x, y, y', y'' also form a G_1.

Similarly, it may be shown that the *thrice-extended* transformations of a given G_1 in the variables x, y, y', y'', y''' form a G_1, and so on to the n-times extended transformations of the G_1.

48. We shall now give a method for finding the *infinitesimal* transformation of an extended G_1, since the conditions for the existence of such a transformation, Art. 26, are obviously fulfilled.

If the finite transformations of the G_1 be given by the equations

$$x_1=\phi(x, y, t), \quad y_1=\psi(x, y, t), \text{ ....(9)}$$

and the infinitesimal transformation by

$$Uf\equiv\xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y},$$

we know, Art. 29, that ξ and η have the forms

$$\frac{\delta x}{\delta t}=\xi, \quad \frac{\delta y}{\delta t}=\eta, \text{ ....(10)}$$

where symbol δ is equivalent to the symbol of differentiation. Thus, since

$$y'=\frac{dy}{dx},$$

we have

$$\frac{\delta y'}{\delta t}=\frac{\delta\frac{dy}{dx}}{\delta t}=\frac{dx\frac{\delta dy}{\delta t}-dy\,\frac{\delta dx}{\delta t}}{dx^2}.$$

Since the order of the operations indicated by δ and d can be reversed,

$$\frac{\delta y'}{\delta t}=\frac{dx\,.\,d\frac{\delta y}{\delta t}-dy\,.\,d\frac{\delta x}{\delta t}}{dx^2};$$

or from (10),

$$\frac{\delta y'}{\delta t}\equiv\frac{d\eta}{dx}-\frac{dy}{dx}\,\frac{d\xi}{dx}\equiv\frac{d\eta}{dx}-y'\cdot\frac{d\xi}{dx}.$$

This is the increment assigned to y' by the transformation (9); we shall usually indicate it by η', so that the infinitesimal transformation of the once-extended G_1 has the form

$$U'f \equiv \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'}.$$

49. In an entirely analogous manner it may be shown that the increment which y'' receives under the transformation of the G_1 (9) has the form

$$\eta'' \equiv \frac{d\eta'}{dx} - y'' \cdot \frac{d\xi}{dx};$$

and generally, the increment of $y^{(n)}$ is

$$\eta^{(n)} \equiv \frac{d\eta^{(n-1)}}{dx} - y^{(n)} \cdot \frac{d\xi}{dx}.$$

Thus the infinitesimal transformation of the n-times extended G_1 is

$$U^{(n)}f \equiv \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'} + \dots + \eta^{(n)}\frac{\partial f}{\partial y^n}.$$

EXAMPLES.

In examples (1)-(9) below, it is required: (*a*) to find by Art. 30 the infinitesimal transformation, Uf, from the accompanying finite transformation, t being the parameter of the G_1; (*b*) conversely, to find the finite transformations of the G_1, by Art. 36 or Art. 29, considering the infinitesimal transformation as being given; (*c*) to find, by Art. 39, what points or lines, if any, are absolutely invariant under each G_1; (*d*) to find, by Art. 37, an invariant of each G_1; (*e*) and, finally, by Arts. 29 and 38, to draw a figure representing the path-curves along which the points of the plane are moved by means of the transformations of each of the G_1 respectively.

(1) $x_1 = x + t,\ y_1 = y;\ Uf \equiv \frac{\partial f}{\partial x}.$

This is the G_1 of translations of all points of the plane, through a distance t, in the direction of the x-axis.

(2) $x_1 = x,\ y_1 = y + t;\ Uf \equiv \frac{\partial f}{\partial y}.$

The G_1 of translations along the y-axis.

(3) $x_1 = tx,\ y_1 = y;\quad Uf \equiv x\dfrac{\partial f}{\partial x}.$

This is a G_1 of so-called *affine* transformations. The effect of performing these transformations upon the points of the plane, when t is positive, is equivalent to a *stretching* of the plane, as if it were a homogeneous elastic plate, in the direction of the x-axis.

(4) $x_1 = tx,\ y_1 = ty;\ Uf \equiv x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y}.$

This is a G_1 of so-called similitudinous transformations. All coordinates are seen to be increased, or diminished, from the origin out, in the same ratio. Hence, a figure in the plane always remains similar to itself under this G_1.

(5) $x_1 = x + \dfrac{tx}{\sqrt{x^2+y^2}},\ y_1 = y + \dfrac{ty}{\sqrt{x^2+y^2}};\ Uf \equiv \dfrac{1}{\sqrt{x^2+y^2}}\left(x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y}\right).$

By means of the finite transformations of this G_1, all points are moved along their radii vectores through the same distance t.

(6) $x_1 = tx,\ y_1 = \dfrac{1}{t}y;\ Uf \equiv x\dfrac{\partial f}{\partial x} - y\dfrac{\partial f}{\partial y}.$

(7) $x_1 = x\cos t - y\sin t,\ y_1 = x\sin t + y\cos t;\ Uf \equiv -y\dfrac{\partial f}{\partial x} + x\dfrac{\partial f}{\partial y}.$

(8) $x_1 = \dfrac{x}{1-tx},\ y_1 = \dfrac{y}{1-tx};\ Uf \equiv x^2\dfrac{\partial f}{\partial x} + xy\dfrac{\partial f}{\partial y}.$

(9) $x_1 = \dfrac{x}{1-ty},\ y_1 = \dfrac{y}{1-ty};\ Uf \equiv xy\dfrac{\partial f}{\partial x} + y^2\dfrac{\partial f}{\partial y}.$

(10) Show that the family of all ∞^2 conic sections whose axes coincide with the coordinate axes,

$$\omega \equiv \frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1,$$

is invariant under the G_1 of affine transformations,

$$x_1 = tx,\quad y_1 = y.$$

Verify the result by making use of the condition, Art. 40,

$$U(\omega) \equiv \Omega(\omega).$$

(11) Show that the family of ∞^1 concentric circles,

$$x^2 + y^2 = r^2,$$

is invariant under the G_1 of similitudinous transformations,

$$x_1 = tx,\quad y_1 = ty,$$

and verify the result by Art. 40.

(12) Show that the family of ∞^2 straight lines,

$$\frac{x}{a}+\frac{y}{b}=1,$$

is invariant under each of the G_1 given in examples (1)-(4) and (6)-(9); that is, that these G_1 are *projective.* Verify the results, as usual, by Art. 40.

(13) (*a*) Show that the family of ∞^2 circles with radius 1,

$$(x-a)^2+(y-b)^2=1,$$

is invariant under the G_1 of rotations given in example (7).

(*b*) Show the same of the family of ∞^1 tangents to the circle,

$$x^2+x^2=1.$$

[See Ex. (17), Chapter I.]

(14) Show that the family of ∞^1 circles,

$$(x-a)^2+y^2=1,$$

is invariant under the G_1 of translations,

$$x_1=x+t, \ y_1=y.$$

(15) Show that the family of ∞^1 circles which touch both axes of coordinates,

$$(x-a)^2+(y-a)^2=a^2,$$

is invariant under the G_1, $x_1=tx$, $y_1=ty$, verifying as usual.

CHAPTER IV.

CONNECTION BETWEEN EULER'S INTEGRATING FACTOR AND LIE'S INFINITESIMAL TRANSFORMATION.

50. We are now prepared to show to what the developments of the preceding Chapters have been tending. In the first section of this Chapter, we shall show how to integrate the *exact* differential equation of the first order in two variables. In the second section, we shall show that a differential equation of the first order in two variables which is *invariant* under a known G_1 may always be integrated by a quadrature; while in the third section, we shall establish some of the most important types of such invariant equations.

SECTION I.

Exact Equations of the First Order. Integrating Factors.

51. A differential equation of the form

$$d\Phi(x, y) \equiv \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy = 0, \qquad \text{.............. (1)}$$

since it is obtained by the *complete* differentiation of an equation of the form

$$\Phi(x, y) = \text{const.},$$

it is said to be an *exact* differential equation; and the first member of (1) is called a *complete differential.*

It is obvious that not every differential equation of the first order,

$$X(x, y)dy - Y(x, y)dx = 0, \quad \ldots\ldots\ldots\ldots\ldots(2)$$

is *exact*; for, to be exact, it is necessary that the condition

$$X \equiv \frac{\partial\Phi}{\partial y}, \quad Y \equiv -\frac{\partial\Phi}{\partial x}$$

be fulfilled. But from this follows

$$\frac{\partial X}{\partial x} = -\frac{\partial Y}{\partial y}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

since each of these quantities must be an expression for

$$\frac{\partial^2\Phi}{\partial y \partial x}.$$

We shall see that this necessary condition that (2) shall be an exact equation is also sufficient. For the most general function, Φ, which satisfies

$$\frac{\partial\Phi}{\partial x} \equiv -Y(x, y),$$

is obtained from

$$\Phi \equiv -\int Y(x, y)dx + Z(y);$$

the integration being performed as if y were a constant, and Z being a function of y alone, which occupies the place of the constant of integration. The only other condition to be satisfied is that the partial differential of Φ with respect to y shall be equal to $X(x, y)$; that is,

$$X(x, y) \equiv \frac{\partial}{\partial y}\left\{-\int Y(x, y)dx + Z(y)\right\}, \quad \ldots\ldots\ldots(4)$$

or

$$\frac{\partial Z}{\partial y} \equiv X(x, y) + \frac{\partial}{\partial y}\int Y(x, y)dx. \quad \ldots\ldots\ldots\ldots(5)$$

Since Z is free of x, the second member of this identity must also be free of x; that is, its partial differential with respect to x must be zero. Hence

$$\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\equiv 0,$$

or

$$\frac{\partial X}{\partial x}\equiv -\frac{\partial Y}{\partial y},$$

which is exactly the condition (3); that is, (3) *is a necessary and a sufficient condition that the differential equation* (2) *shall be exact.*

From the above it follows that the integral of the exact equation (2) may be found by quadrature in the form

$$\Phi\equiv -\int Y(x,\,y)dx+\int\left(X(x,\,y)+\frac{\partial\int Y(x,\,y)dx}{\partial y}\right)dy=\text{const.};$$

or, if more convenient, the equivalent formula,

$$\Phi\equiv -\int X(x,\,y)dy+\int\left(Y(x,\,y)+\frac{\partial\int X(x,\,y)dy}{\partial x}\right)dx=\text{const.},$$

may be used. Here the integration with respect to y is to be performed as if x were a constant; and with respect to x as if y were a constant.

It may be remarked that the equations of Chap. II., Sec. II., are a special class of *exact* equations.

Example 1. In the case of the differential equation

$$(y^2-4xy-2x^2)dy+(x^2-4xy-2y^2)dx=0,$$

the condition (3) is satisfied. For

$$X\equiv y^2-4xy-2x^2,\quad Y\equiv -(x^2-4xy-2y^2),$$

whence, as may be at once verified,

$$\frac{\partial X}{\partial x}=-\frac{\partial Y}{\partial y};$$

so that the differential equation is *exact.*

Using the first of the above formulae for Φ, we find

$$-\int Y(x, y)dx \equiv \frac{x^3}{3} - 2x^2y - 2xy^2;$$

and hence

$$\frac{\partial\int Y(x, y)dx}{\partial y} \equiv 2x^2 + 4xy.$$

Thus

$$X(x, y) + \frac{\partial\int Y(x, y)dx}{\partial y} \equiv y^2 - 4xy - 2x^2 + 2x^2 + 4xy \equiv y^2.$$

Hence

$$\int\left(X(x, y) + \frac{\partial\int Y(x, y)dx}{\partial y}\right)dy \equiv \frac{y^3}{3};$$

so that

$$\Phi(x, y) \equiv \frac{x^3}{3} - 2x^2y - 2xy^2 + \frac{y^3}{3} = \text{const.}$$

is the general integral sought.

It may be readily verified that the use of the second formula for Φ would lead to the same result.

Example 2. Given

$$\frac{2y}{x}dy - \left(1 + \frac{y^2}{x^2}\right)dx = 0.$$

We see that the condition (3) is here satisfied. The second formula for Φ may be used advantageously in this case. We have

$$-\int X\,dy \equiv -\frac{y^2}{x};$$

thus

$$\frac{\partial\int X\,dy}{\partial x} \equiv -\frac{y^2}{x^2}.$$

Hence

$$Y + \frac{\partial\int X\,dy}{\partial x} \equiv 1 + \frac{y^2}{x^2} - \frac{y^2}{x^2} \equiv 1;$$

and the integral of the last expression with respect to x is therefore x. Hence the general integral sought is

$$-\frac{y^2}{x} + x = \text{const.},$$

or

$$x^2 - y^2 = cx. \qquad (c = \text{const.})$$

52. It will usually *not* be the case that the functions X and Y in (2) satisfy the condition (3). But since every differential equation of the first order of the form (2) must have an integral of the general form

$$\Omega(x, y) = \text{const.}, \qquad\qquad (6)$$

the equation

$$\frac{\partial\Omega}{\partial x}dx + \frac{\partial\Omega}{\partial y}dy = 0$$

must be *equivalent* to (2). That is, there must always exist a function $M(x, y)$, such that we can write

$$\frac{\partial\Omega}{\partial x}dx + \frac{\partial\Omega}{\partial y}dy \equiv M(x, y)(X\,dy - Y\,dx);$$

and since the left-hand member of this identity is a complete differential, the right-hand member must be a complete differential also. From (3) we see that M, X, and Y must satisfy the condition

$$\frac{\partial MX}{\partial x} + \frac{\partial MY}{\partial y} \equiv 0.$$

The factor M, which converts the equation (2) into an *exact* differential equation, is called, after its discoverer Euler, an Euler's *integrating factor* of the differential equation (2).

Example. In the equation

$$(x - yx^2)dy + (y + xy^2)dx = 0,$$

the condition (3) is *not* satisfied. Hence this equation is not exact. If the equation be multiplied, however, by

$$M \equiv \frac{1}{x^2y^2},$$

it will become exact; and the method of the preceding article gives as the general integral,

$$\log\frac{x}{y} - \frac{1}{xy} = \text{const.}$$

SECTION II.

A Differential Equation of the First Order, which is Invariant under a known G_1, may be integrated by a Quadrature.

53. Having seen in the last section that an *exact* differential equation of the first order in two variables may be integrated by a quadrature, and that the knowledge of an integrating factor of a given differential equation, which is *not* exact, enables us to put the equation into an exact form, we shall show in this section what it means for a differential equation of the first order to be *invariant* under a given G_1; and we shall see that such an invariant equation may be integrated by a quadrature.

54. In order that an algebraic equation

$$\omega(x, y, y')=0,$$

in the three variables x, y, y' may be invariant under a given G_1, in the same variables,

$$U'f\equiv\xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y}+\eta'\frac{\partial f}{\partial y'},$$

it is, by Art. 42, a necessary and sufficient condition that the expression $U'(\omega)$ shall be zero, either identically or by means of $\omega=0$. It was also shown, Art. 43, that if u and v are two independent solutions of the linear partial differential equation

$$U'f\equiv\xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y}+\eta'\frac{\partial f}{\partial y'}=0,$$

the most general form of the invariant equation $\omega=0$ is

$$\Omega(u, v)=0, \text{ or } u-\mathrm{F}(v)=0.$$

55. If now y' be considered the differential coefficient of y with respect to x, the equation

$$\omega(x, y, y')=0$$

will be a differential equation of the first order; and if we consider $U'f$ to be the once-extended G_1 corresponding to the G_1 in two variables,

$$Uf \equiv \xi(x, y)\frac{\partial f}{\partial x} + \eta(x, y)\frac{\partial f}{\partial y},$$

when the expression $U'(\omega)$ is zero, either identically or by means of the equation $\omega=0$, *the differential equation of the first order*, $\omega=0$, *is said to be invariant under, or to admit of, the* G_1,

$$U'f \equiv \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'}.$$

Also we see that *to find the most general differential equation of the first order which shall be invariant under a given* G_1, $U'f$, *it is necessary to find two independent solutions of the linear partial differential equation of the first order,*

$$U'f \equiv \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'} = 0;$$

that is to say, we must find two independent integral-functions of the simultaneous system

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dy'}{\eta'}.$$

One of these integral-functions may be found from the equation

$$\frac{dx}{\xi} = \frac{dy}{\eta};$$

and since ξ and η are free of y', this integral-function, which we shall call u, will not contain y'. The second integral-function, which we shall denote by v, and for finding which one method has been indicated, Chap. II., Sec. 2, must contain y'. The most general invariant differential equation will then have the form

$$v - \mathrm{F}(u) = 0.$$

56. To find the integral-function u of the preceding

article, it is theoretically necessary to integrate a differential equation of the first order, namely,

$$\frac{dx}{\xi}=\frac{dy}{\eta}.$$

But from the form of this equation, we know, Art. 38, that $u=$const. must represent the path-curves of the G_1,

$$Uf\equiv\xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y}.$$

Hence, if the path-curves of the G_1, Uf, are known, of course u is also known; and it will be remembered, Chap. III., Examples, that the path-curves of a large number of the most important G_1's in the plane can be found by integrating differential equations of the first order which are *exact*. Thus, in a large number of the most important cases, u can be found by a quadrature.

We propose to show now that *if u has been found, v can be found by a quadrature.*

We have already seen, Art. 43, that every infinitesimal transformation in n variables can be brought, by a proper choice of variables, to the form of a mere translation. If u be known, we shall first show that in this case

$$Uf\equiv\xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y}$$

can be brought to the form of a mere translation *by a quadrature.*

Let us introduce into Uf the new variables x_1, y_1; and demand that Uf assume the form of a translation. Thus Uf, Art. 35, becomes

$$U_1f\equiv U(x_1)\frac{\partial f}{\partial x_1}+U(y_1)\frac{\partial f}{\partial y_1}.$$

In order that U_1f shall have the form of the translation $\frac{\partial f}{\partial y_1}$ in the new variables, it is necessary to have

$$U(x_1)\equiv\xi\frac{\partial x_1}{\partial x}+\eta\frac{\partial x_1}{\partial y}\equiv 0\,;\ U(y_1)\equiv\xi\frac{\partial y_1}{\partial x}+\eta\frac{\partial y_1}{\partial y}\equiv 1.$$

That is to say, x_1 must be a solution of the partial equation $Uf=0$;

and since u is a solution of this equation, being by hypothesis the integral-function of the ordinary differential equation

$$\frac{dx}{\xi}=\frac{dy}{\eta},$$

we may assume $x_1 \equiv u$.

Now y_1 must be a function of x and y, which satisfies the equation

$$\xi\frac{\partial y_1}{\partial x}+\eta\frac{\partial y_1}{\partial y}=1,$$

and we may assume that y_1, x, and y are connected by an equation of the general form

$$\Omega(x, y, y_1)=\text{const.}$$

By differentiating this equation with respect to x and with respect to y successively, we find

$$\frac{\partial\Omega}{\partial x}+\frac{\partial\Omega}{\partial y_1}\frac{\partial y_1}{\partial x}=0,$$

$$\frac{\partial\Omega}{\partial y}+\frac{\partial\Omega}{\partial y_1}\frac{\partial y_1}{\partial y}=0.$$

Multiplying the first equation by ξ and the second by η, and adding, we obtain

$$\xi\frac{\partial\Omega}{\partial x}+\eta\frac{\partial\Omega}{\partial y}+\frac{\partial\Omega}{\partial y_1}\left(\xi\frac{\partial y_1}{\partial x}+\eta\frac{\partial y_1}{\partial y}\right)=0;$$

or, on account of the differential equation connecting x, y, and y_1,

$$\xi\frac{\partial\Omega}{\partial x}+\eta\frac{\partial\Omega}{\partial y}+\frac{\partial\Omega}{\partial y_1}=0.$$

But, by Art. 18, this linear partial differential equation is equivalent to the simultaneous system

$$\frac{dx}{\xi}=\frac{dy}{\eta}=\frac{dy_1}{1};$$

that is to say, y_1 may be found as a function of x and y by integrating this simultaneous system in the three variables x, y, y_1. But we already know one integral-function of the system, namely, x_1 or u. Hence it is obvious that y_1 may be found by a quadrature; for we only need to eliminate, Art. 23, say x out of the equation

$$\frac{dy}{\eta}=dy_1$$

by means of $u=\text{const.}$, when we have an ordinary differential equation between y and y_1, in which the variables are separate.

Thus, by a quadrature, we have found the new variables which make Uf take the form of a mere translation, $\frac{\partial f}{\partial y_1}$. But the differential equations which are invariant under this translation are easily found. The extended G_1 in the variables x_1, y_1, evidently has the form

$$U'_1 f \equiv \frac{\partial f}{\partial y_1},$$

since, Art. 48,

$$\eta'_1 \equiv \frac{d\eta_1}{dx_1} - y'_1 \frac{d\xi_1}{dx_1} \equiv 0,$$

where ξ_1, η_1, η'_1, and y'_1 have the usual meaning. Hence to find the invariant equations, we must find two integral functions of the simultaneous system

$$\frac{dx_1}{0} = \frac{dy_1}{1} = \frac{dy'_1}{0},$$

since ξ_1 and η'_1 are zero. But x_1 and y'_1 are evidently two independent integral-functions of this system. Hence the general invariant differential equation in the variables x_1, y_1 will have the form

$$\Omega(x_1, y'_1) = 0.$$

If, now, we return to our former variables, this equation must take the form of a function of u and v equated to zero, say

$$\mathrm{F}(u, v) = 0.$$

But since x_1 is identical with u, y'_1 must be a function of v; and we can obviously assume $y'_1 = v$.

Hence when the path-curves, $u =$ const., of a given G_1 are known, the most general differential equation of the first order which is invariant under the given G_1 may be found by quadratures. Practically the calculations may usually be made much shorter than indicated above, since in the most important cases the variables in the simultaneous system to be integrated,

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dy'}{\eta'},$$

may be separated by inspection.

57. In Art. 37 the function u, which is a solution of the linear partial differential equation

$$Uf \equiv \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} = 0,$$

was called an *invariant* of the G_1, Uf. Similarly, the function v, which we saw must always contain y', and which is a solution of

$$U'f \equiv \xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y}+\eta'\frac{\partial f}{\partial y'}=0,$$

is called a *differential invariant of the first order* of the G_1, Uf.

58. If x, y, and y' be considered the coordinates of a lineal element in the plane, the equation

$$\omega(x, y, y')=0 \text{ ....}(1)$$

represents, Art. 44, ∞^2 of the ∞^3 lineal elements; and to demand that the equation $\omega=0$ shall be invariant under the G_1, $U'f$, is the same as to demand that the family of ∞^2 lineal elements shall, as a whole, be invariant under $U'f$. For, the analytical criterion that (1) shall be invariant, means, interpreted geometrically, that the transformed (1) shall represent *the same* family of ∞^2 lineal elements that (1) itself does. But these ∞^2 lineal elements envelop the ∞^1 integral curves of (1), considering this equation as an ordinary differential equation of the first order; and since the family of lineal elements is invariant, the family of ∞^1 integral curves must also be invariant under the G_1, $U'f$.

Thus, if

$$\Phi(x, y)=\text{const. ....}(2)$$

represent these integral curves, since (2), which does not contain y' at all, must be invariant under the extended G_1, $U'f$, this equation must also be invariant under the G_1, Uf; that is, by Art. 40, a condition of the form

$$U\big(\Phi(x, y)\big) \equiv W(\Phi) \text{ ....}(3)$$

must hold, if the differential equation (1) is invariant under $U'f$.

Conversely, if a condition of the form (3) holds, of course the ∞^1 integral curves (2) are invariant—and with them, the family of ∞^2 lineal elements (1)—or, as we may say, the differential equation of the first order (1) is invariant under $U'f$. If, therefore, a G_1 is known, of which the integral curves of a given differential equation of the first order admit, this equation, written in the form (1), always admits of the extended G_1.

Hence, we may also define an ordinary differential equation of the first order as being invariant under a given G_1, Uf, when an integral-function Φ of that equation is transformed by means of Uf into a function which is itself an integral-function of the differential equation; that is, when a relation of the form (3) exists.

59. We shall now show that a differential equation of the first order in two variables, which is invariant under a known G_1, may be integrated by a quadrature.

Let the given differential equation be

$$\Omega(x, y, y') = 0; \quad \text{.......................(4)}$$

and suppose (4) to admit of the G_1,

$$U'f \equiv \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'}. \quad \text{.................(5)}$$

We shall, for reasons explained in Art. 60, assume that $\Omega = 0$ is *not* the differential equation of the ∞^1 path-curves of the G_1, Uf.

If (4) be written in the solved form

$$X(x, y)dy - Y(x, y)dx = 0, \quad \text{..............(6)}$$

and if its integral-function be designated by $\omega(x, y)$, by Art. 16, ω must be a solution of the linear partial differential equation of the first order,

$$X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial y} = 0. \quad \text{.......................(7)}$$

Moreover, since the family of integral curves $\omega = \text{const.}$ is invariant, it follows from Arts. 40 and 58 that

$$U(\omega) \equiv \xi\frac{\partial \omega}{\partial x} + \eta\frac{\partial \omega}{\partial y} \equiv W\big(\omega(x, y)\big), \quad \text{...........(8)}$$

Now if $\Phi(\omega)$ be a certain function of ω alone, Φ will also be an integral-function of (6), and $U(\Phi)$ will depend upon Φ *alone.* For

$$U(\Phi) \equiv \frac{d\Phi}{d\omega} U(\omega) \equiv \frac{d\Phi}{d\omega} W(\omega).$$

and ω may be removed from the right-hand member of the last identity by means of

$$\Phi \equiv \Phi(\omega),$$

giving thus $U(\Phi)$ as a function of Φ alone.

Since we assumed above that the curves $\omega = c$ were not the always invariant *path-curves* of the G_1, Uf, the function $W(\omega)$ in (8) cannot be zero; and we may easily choose Φ as such a function of ω that $U(\Phi) \equiv 1$. For it is only necessary to determine Φ so that

$$\frac{d\Phi}{d\omega} \cdot W(\omega) = 1,$$

or

$$\Phi = \int \frac{d\omega}{W(\omega)}.$$

Since $\Phi = \text{const.}$ represents the same family of curves that $\omega = \text{const.}$ does, let us suppose ω so chosen from the beginning that $U(\omega) \equiv 1$; that is, let us now designate by ω the function which we have just called Φ. Then we have

$$X\frac{\partial \omega}{\partial x} + Y\frac{\partial \omega}{\partial y} \equiv 0,$$

and

$$U(\omega) \equiv \xi\frac{\partial \omega}{\partial x} + \eta\frac{\partial \omega}{\partial y} \equiv 1.$$

Hence

$$\frac{\partial \omega}{\partial x} \equiv \frac{-Y}{X\eta - Y\xi}, \quad \frac{\partial \omega}{\partial y} \equiv \frac{X}{X\eta - Y\xi};$$

that is,

$$d\omega \equiv \frac{\partial \omega}{\partial x} dx + \frac{\partial \omega}{\partial y} dy \equiv \frac{X\,dy - Y\,dx}{X\eta - Y\xi}.$$

Since the first member of the last equation is necessarily a complete differential, the same must be true of the second member; that is, we have the

Theorem.[*] *If a given differential equation of the first order in two variables*

$$X\,dy - Y\,dx = 0$$

admits of a known G_1,

$$Uf \equiv \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y},$$

whose path-curves are not identical with the integral curves of the differential equation, then

$$M \equiv \frac{1}{X\eta - Y\xi}$$

is an integrating factor of the differential equation; and the general integral may be found by a quadrature in the form

$$\int\frac{X\,dy - Y\,dx}{X\eta - Y\xi} = \text{const.}$$

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[*] This theorem was first published by Lie in the "Verhandlungen der Gesellschaft der Wissenschaften zu Christiania," November, 1874.

By Art. 52 the equation

$$X\,dy - Y\,dx = 0$$

always possesses an integrating factor, M; and if M be known, it follows from the developments in the text that it is only necessary to choose ξ or η in

$$Uf \equiv \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y}$$

in such manner that

$$\frac{1}{X\eta - Y\xi} \equiv M,$$

when the given differential equation will be *invariant* under Uf. Although it follows from this that *every* differential equation of the first order is invariant under an unlimited number of G_1's, when we speak of an *invariant* differential equation in this book, we shall always mean one which is invariant under a *known* G_1.

Example 1. The differential equation

$$\Omega \equiv xy' - y + x^2 = 0$$

admits of the G_1, $Uf \equiv x\frac{\partial f}{\partial y}$. For the extended transformation is found by forming, Art. 48,

$$\eta' \equiv \frac{d\eta}{dx} - y'\frac{d\xi}{dx},$$

which in this case, since $\eta \equiv x$, $\xi \equiv 0$, is 1. Hence

$$U'f \equiv x\frac{\partial f}{\partial y} + \frac{\partial f}{\partial y'}.$$

By Art. 55, the expression $U'(\Omega)$ must be zero, either identically, or by means of $\Omega = 0$. We find

$$U'(\Omega) \equiv x\frac{\partial}{\partial y}(xy' - y + x^2) + \frac{\partial}{\partial y'}(xy' - y + x^2)$$
$$\equiv -x + x \equiv 0.$$

Hence the condition that $\Omega = 0$ shall admit of Uf is satisfied. Now write $\Omega = 0$ in the solved form,

$$x\,dy - (y - x^2)dx = 0\,;$$

since this equation admits of $x\frac{\partial f}{\partial y}$, the integrating factor

$$M \equiv \frac{1}{X\eta - Y\xi}$$

has in this case the value,

$$\frac{1}{x\,.\,x - (y - x^2)\,.\,0} = \frac{1}{x^2}\,;$$

and the integral is found to be,

$$\int \frac{x\,dy - (y - x^2)dx}{x^2} = \text{const.}$$

or, by Art. 51,

$$\omega = \frac{y + x^2}{x} = \text{const.}$$

We may at once verify that $\omega = \text{const.}$ admits of the infinitesimal transformation of the G_1, $x\frac{\partial f}{\partial y}$; as well as of the finite transformations, $x_1 = x$, $y_1 = y + xt$.

Example 2. The differential equation

$$\Omega \equiv xy\,y' - (x+y^2) = 0$$

admits of the already extended G_1

$$U'f \equiv 2x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} - y'\frac{\partial f}{\partial y'}.$$

For here $U'(\Omega)$ has the form,

$$\begin{aligned} U'(\Omega) & \\ &\equiv 2x\frac{\partial}{\partial x}(xy\,y' - x - y^2) + y\frac{\partial}{\partial y}(xy\,y' - x - y^2) - y'\frac{\partial}{\partial y'}(xy\,y' - x - y^2) \\ &\equiv 2xy\,y' - 2x + xy\,y' - 2y^2 - xy\,y' \equiv 2(xy\,y' - x - y^2) \equiv 2\Omega\,; \end{aligned}$$

and the condition that $\Omega = 0$ shall admit of Uf is satisfied. Now write the differential equation in the form,

$$xy\,dy - (x+y^2)dx = 0\,;$$

since it admits of

$$U'f \equiv 2x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} - y'\frac{\partial f}{\partial y'},$$

an integrating factor must be given by

$$M \equiv \frac{1}{X\eta - Y\xi} \equiv \frac{1}{xy\,y - (x+y^2)\,2x} \equiv \frac{1}{-2x^2 - xy^2}.$$

Hence the integral is

$$\int \frac{xy\,dy - (x+y^2)dx}{2x^2 + xy^2} = \text{const.},$$

or, Art. 51,

$$\frac{2x+y^2}{x^2} = \text{const.}$$

60. The method of integration of Art. 59 fails when

$$X\eta - Y\xi \equiv 0.$$

For this case, we see

$$\frac{X}{Y} \equiv \frac{\xi}{\eta};$$

and since the first of these ratios gives the direction of the tangent to the integral curves of $\Omega = 0$ through the point (x, y), and the second ratio gives the direction in which the point (x, y) is moved by means of the G_1, Uf, the above identity states that the point (x, y) always moves on one of the integral curves of $\Omega=0$. Hence the invariant family of ∞^1 curves is none other than the family of ∞^1 always invariant path-curves of the G_1—each curve being separately invariant. In other words, the G_1, Uf tells us nothing new with regard to the equation $\Omega=0$, and hence Uf is, in this case, said to be *trivial* with respect to that differential equation. In Art. 59 the case that Uf shall be *trivial* is always excluded.

When Uf is trivial, since

$$\frac{Y}{X}\equiv\frac{\eta}{\xi},$$

we may write

$$\xi\equiv\rho(x, y)X,\ \ \eta=\rho(x, y)Y;$$

so that Uf has the form

$$Uf\equiv\rho(x, y)\left(X\frac{\partial f}{\partial x}+Y\frac{\partial f}{\partial y}\right).$$

Thus it is seen that every transformation of the form

$$\rho(x, y)\left(X\frac{\partial f}{\partial x}+Y\frac{\partial f}{\partial y}\right)$$

is *trivial*, with respect to the ordinary differential equation

$$Xdy-Ydx=0.$$

In future, we shall always disregard *trivial* infinitesimal transformations.

SECTION III.

Classes of Differential Equations of the First Order which admit of a given G_1 in Two Variables.

61. Having shown that an ordinary differential equation of the first order in two variables can be integrated by a quadrature when it admits of a known G_1, the next

step will be to find the classes of differential equations of the first order which admit of certain of the simpler G_1 in two variables.

From Art. 58 it is clearly immaterial whether we say that the family of ∞^1 integral curves of the given differential equation is invariant under a G_1, Uf, or whether we say that the differential equation itself is invariant under the G_1, Uf, or under the equivalent once-extended G_1, $U'f$.

62. *To find all differential equations of the first order which admit of a translation along the x-axis.*

This translation is represented, Example 1, Chapter III., by

$$Uf \equiv \frac{\partial f}{\partial x}.$$

We see at once, that since $\xi \equiv 1$ and $\eta \equiv 0$,

$$\eta' \equiv \frac{d\eta}{dx} - y'\frac{d\xi}{dx} \equiv 0;$$

that is,
$$U'f \equiv \frac{\partial f}{\partial x}.$$

To find the most general invariant differential equation, we must, Art. 55, find two independent integral-functions of the simultaneous system

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dy'}{0}.$$

It is evident that y and y' may be chosen as the functions designated as u and v in Art. 55; and hence the most general differential equation of the first order which admits of a translation along the x-axis has the form

$$\Omega(y, y') = 0;$$

or, if solved in terms of y',

$$y' - \mathrm{F}(y) = 0.$$

In this equation the variables are separate, so that the integration may be accomplished by a quadrature.

Analogously, it is obvious that all differential equations of the first order which admit of the G_1 of translations along the y-axis

$$Uf \equiv \frac{\partial f}{\partial y}$$

have the form

$$y' - \mathrm{F}(x) = 0;$$

and are immediately integrable by quadrature.

63. *To find all differential equations of the first order which admit of the G_1 of affine transformations*

$$Uf \equiv x\frac{\partial f}{\partial x}.$$

Here, since $\xi \equiv x$ and $\eta \equiv 0$,

$$\eta' \equiv \frac{d\eta}{dx} - y'\frac{d\xi}{dx} \equiv -y'.$$

Hence the extended G_1 is

$$U'f \equiv x\frac{\partial f}{\partial x} - y'\frac{\partial f}{\partial y'};$$

and the simultaneous system to be integrated is

$$\frac{dx}{x} = \frac{dy}{0} = \frac{dy'}{-y'}.$$

One integral-function is evidently y; and, from

$$\frac{dx}{x} + \frac{dy'}{y'} = 0,$$

a second is found to be xy'.

Hence the most general invariant differential equation of the first order has the form

$$\Omega(xy', y) = 0;$$

or

$$xy' - \mathrm{F}(y) = 0.$$

Here again the variables are separate, so that the equation may at once be integrated by a quadrature.

The general form of the differential equations which are invariant under the corresponding G_1 of affine transformations along the y-axis,

$$Uf \equiv y\frac{\partial f}{\partial y},$$

is readily seen to be

$$y' - y\mathrm{F}(x) = 0.$$

In this equation also the variables may be separated by inspection.

64. *To find all differential equations of the first order which admit of the G_1,*

$$Uf \equiv x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}.$$

Here, since $\xi \equiv x$ and $\eta \equiv y$, we find in the usual manner $\eta' \equiv 0$. Hence the simultaneous system to be integrated is

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dy'}{0}.$$

The integral-functions of this system, usually designated by u and v, are obviously

$$\frac{y}{x},\ y'.$$

Hence, the most general differential equation of the first order which is invariant under the G_1

$$Uf \equiv x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$$

has the form

$$\Omega\left(y', \frac{y}{x}\right) = 0;$$

In this equation the variables are separate, so that the integration may be accomplished by a quadrature.

Analogously, it is obvious that all differential equations of the first order which admit of the G_1 of translations along the y-axis

$$Uf \equiv \frac{\partial f}{\partial y}$$

have the form

$$y' - \mathrm{F}(x) = 0;$$

and are immediately integrable by quadrature.

63. *To find all differential equations of the first order which admit of the G_1 of affine transformations*

$$Uf \equiv x\frac{\partial f}{\partial x}.$$

Here, since $\xi \equiv x$ and $\eta \equiv 0$,

$$\eta' \equiv \frac{d\eta}{dx} - y'\frac{d\xi}{dx} \equiv -y'.$$

Hence the extended G_1 is

$$U'f \equiv x\frac{\partial f}{\partial x} - y'\frac{\partial f}{\partial y'};$$

and the simultaneous system to be integrated is

$$\frac{dx}{x} = \frac{dy}{0} = \frac{dy'}{-y'}.$$

One integral-function is evidently y; and, from

$$\frac{dx}{x} + \frac{dy'}{y'} = 0,$$

a second is found to be xy'.

Hence the most general invariant differential equation of the first order has the form

$$\Omega(xy', y) = 0;$$

or

$$xy' - \mathrm{F}(y) = 0.$$

Here again the variables are separate, so that the equation may at once be integrated by a quadrature.

The general form of the differential equations which are invariant under the corresponding G_1 of affine transformations along the y-axis,

$$Uf \equiv y\frac{\partial f}{\partial y},$$

is readily seen to be

$$y' - y\mathrm{F}(x) = 0.$$

In this equation also the variables may be separated by inspection.

64. *To find all differential equations of the first order which admit of the* G_1,

$$Uf \equiv x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}.$$

Here, since $\xi \equiv x$ and $\eta \equiv y$, we find in the usual manner $\eta' \equiv 0$. Hence the simultaneous system to be integrated is

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dy'}{0}.$$

The integral-functions of this system, usually designated by u and v, are obviously

$$\frac{y}{x}, \; y'.$$

Hence, the most general differential equation of the first order which is invariant under the G_1

$$Uf \equiv x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$$

has the form

$$\Omega\left(y', \frac{y}{x}\right) = 0;$$

or, when solved in terms of y',

$$y' - \mathrm{F}\left(\frac{y}{x}\right) = 0,$$

This is the so-called *general homogeneous* equation of the first order.

We may write the above equation in the form

$$dy - \mathrm{F}\left(\frac{y}{x}\right)dx = 0;$$

and the method of Art. 59, gives

$$M \equiv \frac{1}{y - x\mathrm{F}\left(\frac{y}{x}\right)}$$

as an integrating factor. Hence the equation written in the form

$$\frac{dy - \mathrm{F}\left(\frac{y}{x}\right)dx}{y - x\mathrm{F}\left(\frac{y}{x}\right)} = 0$$

is *exact*, and may be integrated, by the method of Art. 51, by a quadrature.

Example. Given

$$(3x^2y + 2y^3)dy + x^3dx = 0.$$

This equation, being homogeneous, belongs to the class of the present article.

Written in the form

$$dy - \mathrm{F}\left(\frac{y}{x}\right)dx = 0,$$

it becomes,
$$dy + \frac{x^3}{3x^2y + 2y^3}dx = 0;$$

so that
$$\frac{dy + \frac{x^3}{3x^2y + 2y^3}dx}{y + x\frac{x^3}{3x^2y + 2y^3}} = 0,$$

or
$$\frac{3x^2y+2y^3}{x^4+3x^2y^2+2y^4}dy+\frac{x^3}{x^4+3x^2y^2+2y^4}dx=0,$$

must be an *exact* equation. It may easily be verified that such is the case; and the general integral is found by Art. 51 to be

$$\frac{x^2+2y^2}{\sqrt{x^2+y^2}}=\text{const.}$$

65. It should be noticed that equations of the general form

$$(ax+by+c)dx-(a'x+b'y+c')dy=0,\quad (a, \ldots, c'=\text{const.})$$

may usually be made homogeneous by a proper choice of variables. For, let the new variables be

$$\bar{x}=x-h,\quad \bar{y}=y-k,\qquad (h, k=\text{const.})$$

then the given equation becomes

$$(a\bar{x}+b\bar{y}+ah+bk+c)d\bar{x}-(a'\bar{x}+b'\bar{y}+a'h+b'k+c')d\bar{y}=0.$$

If, now, h and k are determined from the equations

$$ah+bk+c=0,$$
$$a'h+b'k+c'=0,$$

the above equation in $\bar{x}$ and $\bar{y}$ will evidently become homogeneous, and thus may be integrated by the method of the preceding article.

This method fails when $a:a'=b:b'$. Let us assume then

$$a=n\,.\,a',\quad b=n\,.\,b',\qquad (n=\text{const.})$$

and the original equation becomes

$$(ax+by+c)\,dx-\{n(ax+by)+c'\}\,dy=0.$$

Now introduce in place of y the new variable

$$z=ax+by;$$

and it is readily seen that the differential equation takes the form

$$\frac{dz}{dx}=a+b\,\frac{z+c}{nz+c'},$$

in which the variables may be separated by inspection.

Example. Given $(2y-x-1)dy+(2x-y+1)dx=0$.

Here the equations

$$ah+bk+c=0,$$
$$a'h+b'k+c'=0,$$

have the forms

$$2h-k+1=0,$$
$$-h+2k-1=0;$$

so that

$$h=-\tfrac{1}{3},\quad k=\tfrac{1}{3}.$$

Introducing the new variables

$$\bar{x}=x+\tfrac{1}{3},\quad \bar{y}=y-\tfrac{1}{3},$$

the given differential equation becomes

$$(2\bar{y}-\bar{x})d\bar{y}+(2\bar{x}-\bar{y})d\bar{x}=0.$$

The general integral of this homogeneous equation is found to be

$$\bar{x}^2-\bar{x}\bar{y}+\bar{y}^2=\text{const.};$$

and if we now return to the original variables, the general integral of the given differential equation is seen to be

$$x^2-xy+y^2+x-y=\text{const.}$$

66. *To find all differential equations of the first order which admit of the G_1 of rotations.*

A rotation around the origin is given, as will be remembered, by the G_1,

$$Uf\equiv -y\frac{\partial f}{\partial x}+x\frac{\partial f}{\partial y}.$$

Here $\xi\equiv -y$, $\eta\equiv x$; and hence

$$\eta'\equiv\frac{d\eta}{dx}-y'\frac{d\xi}{dx}\equiv 1+y'^2.$$

It is necessary, therefore, to find two integral-functions of the simultaneous system

$$\frac{dx}{-y}=\frac{dy}{x}=\frac{dy'}{1+y'^2}.$$

From the first equation, which may be written,

$$xdx+ydy=0,$$

we see that one integral function is

$$u \equiv x^2 + y^2.$$

By the method of Art. 23, we now write

$$x^2 + y^2 = c^2, \qquad (c^2 = \text{const.})$$

whence

$$x = \sqrt{c^2 - y^2},$$

and

$$\frac{dy}{\sqrt{c^2 - y^2}} - \frac{dy'}{1 + y'^2} = 0.$$

The variables are separate: hence by immediate integration

$$\sin^{-1}\frac{y}{c} - \tan^{-1} y' = b; \qquad (b = \text{const.})$$

or,

$$\sin^{-1}\frac{y}{\sqrt{x^2 + y^2}} - \tan^{-1} y' = b;$$

But this may be written

$$\tan^{-1}\frac{y}{x} - \tan^{-1} y' = b;$$

or, taking the tangent of both sides, the second integral is found to be

$$v \equiv \frac{xy' - y}{x + yy'} = \text{const.}$$

Thus the most general invariant differential equation of the first order has the form

$$\frac{xy' - y}{x + yy'} - \mathrm{F}(x^2 + y^2) = 0.$$

This equation may be written—when F is put for $\mathrm{F}(x^2 + y^2)$,

$$(x - y\mathrm{F})dy - (y + x\mathrm{F})dx = 0.$$

The method of Art. 59 gives, as an integrating factor of all equations of this form,

$$U \equiv \frac{1}{x^2 + y^2};$$

so that the above equation, written in the form

$$\frac{(x-y\mathrm{F})dy-(y+x\mathrm{F})dx}{x^2+y^2}=0$$

is *exact*, and may be integrated, by Art. 51, by a quadrature.

67. *To find all differential equations of the first order which admit of the* G_1

$$Uf\equiv x\frac{\partial f}{\partial x}-y\frac{\partial f}{\partial y}.$$

Here η' will be found to have the value $-2y'$; so that the simultaneous system to be integrated has the form

$$\frac{dx}{x}=\frac{dy}{-y}=\frac{dy'}{-2y'}.$$

Since the variables are here separate, it is seen at once that two independent integral-functions are

$$xy,\ \frac{y'}{y^2}.$$

We may write the second integral-function in the form

$$\frac{xy'}{xy\cdot y};$$

and since xy is itself an integral-function, we see that $\frac{xy'}{y}$ must also be an integral-function. Thus we may assume

$$u\equiv xy,\ v\equiv\frac{xy'}{y};$$

so that the most general invariant differential equation of the first order will have the form

$$y'\cdot\frac{x}{y}-\mathrm{F}(xy)=0.$$

We may assume $\mathrm{F} \equiv \frac{f_1(xy)}{f_2(xy)}$, and write the last equation symmetrically

$$f_1(xy) \,.\, x\,dy - f_2(xy) \,.\, y\,dx = 0.$$

Of course all equations of this form may be integrated by a quadrature; since the method of Art. 59 gives as an integrating factor,

$$M \equiv \frac{1}{xy\left(f_1(xy) + f_2(xy)\right)}.$$

Example. Given

$$(x - yx^2)dy + (y + xy^2)dx = 0.$$

This equation may be written

$$(1 - xy)x\,dy + (1 + xy)y\,dx = 0,$$

so that it is seen to belong to the class of the present article. Hence an integrating factor is

$$U \equiv \frac{1}{2x^2y^2};$$

so that the equation

$$\frac{1 - xy}{xy^2}dy + \frac{1 + xy}{x^2y}dx = 0$$

is *exact.* The general integral is found in the usual way to be

$$\log\frac{x}{y} - \frac{1}{xy} = \text{const.}$$

68. *To find all differential equations of the first order which admit of the* G_1

$$Uf \equiv e^{\int \phi(x)dx} \,.\, \frac{\partial f}{\partial y}.$$

Proceeding as usual, we find for η' the value

$$\eta' \equiv \phi(x) e^{\int \phi(x)dx};$$

so that the simultaneous system to be integrated has the form

$$\frac{dx}{0} = \frac{dy}{e^{\int \phi(x)dx}} = \frac{dy'}{\phi(x) \,.\, e^{\int \phi(x)dx}}.$$

One integral-function is evidently

$$u \equiv x.$$

A second may be obtained from

$$\frac{dy}{e^{\int \phi(x)dx}} = \frac{dy'}{\phi(x) \, . \, e^{\int \phi(x)dx}},$$

or

$$dy = \frac{dy'}{\phi(x)}.$$

Since $x = \text{const.}$ is one integral of the simultaneous system, $\phi(x)$ plays the rôle of a mere constant in the last equation. Hence, by a quadrature, a second integral-function is found to be

$$v \equiv y - \frac{y'}{\phi(x)}.$$

The most general invariant differential equation has, therefore, the form

$$\frac{y'}{\phi(x)} - y - \mathrm{F}(x) = 0,$$

or

$$y' - \phi(x)y - \psi(x) = 0.$$

This is the so-called *general linear differential equation of the first order.* Of course all equations of this form may be integrated by a quadrature. For the above equation may be written

$$dy - \{y\phi(x) + \psi(x)\} \, dx = 0\,;$$

and by Art. 59,

$$M \equiv \frac{1}{e^{\int \phi(x)dx}}$$

is an integrating factor, so that

$$e^{-\int \phi(x)dx} \, . \, dy - \{y\phi(x) + \psi(x)\} \, e^{-\int \phi(x)dx} \, . \, dx = 0$$

must be an *exact* equation. By Art. 51 the general integral is found in the form

$$y=e^{\int\phi(x)dx}\left\{\int\psi(x)e^{-\int\phi(x)dx}\,.\,dx+\text{const.}\right\}.$$

In an analogous manner it may be shown that the differential equations of the first order which are invariant under

$$Uf\equiv e^{\int\phi(y)dy}\,.\,\frac{\partial f}{\partial x}$$

have the general form

$$dx-\{\phi(y)x+\psi(y)dy=0.$$

This general equation, which, of course, may be integrated by a quadrature by the usual method, is said to be *linear in x*, y being chosen as the independent variable.

Example. Given

$$y'-\frac{x}{1+x^2}y-\frac{1}{1+x^2}=0.$$

In this linear equation the functions ϕ and ψ have the form

$$\phi\equiv\frac{x}{1+x^2},\ \psi\equiv\frac{1}{1+x^2}.$$

Hence $$e^{\int\phi(x)dx}\equiv\sqrt{1+x^2},\ e^{-\int\phi(x)dx}\equiv\frac{1}{\sqrt{1+x^2}};$$

so that $$\psi(x)e^{-\int\phi(x)dx}\equiv\frac{1}{(1+x^2)^{\frac{3}{2}}};$$

and $$\int\psi(x)e^{-\int\phi(x)dx}dx\equiv\int\frac{dx}{(1+x^2)^{\frac{3}{2}}}\equiv\frac{x}{\sqrt{1+x^2}}+\text{const.}$$

Making these substitutions in the formula for the general integral of the linear equation, we find

$$y=\sqrt{1+x^2}\left\{\frac{x}{\sqrt{1+x^2}}+c\right\},\qquad (c=\text{const.})$$

or $$y=x+c\sqrt{1+x^2},$$

as the general integral required.

69. It should be noticed that equations of the form

$$y' - \phi(x)y - \psi(x)y^n = 0$$

may be easily brought to the linear form. For, dividing by y^n the equation becomes

$$y^{-n} . y' - \phi(x)y^{1-n} - \psi(x) = 0.$$

If now we put $$z = y^{1-n},$$

so that $$y^{-n} . \frac{dy}{dx} = \frac{1}{1-n}\frac{dz}{dx},$$

the above equation becomes

$$\frac{1}{1-n}\frac{dz}{dx} - \phi(x)z - \psi(x) = 0,$$

an equation which is linear in z.

Example. Given

$$y' + \frac{y}{x} - \frac{y^2 \log x}{x} = 0.$$

The equation may be written

$$y^{-2} . y' + \frac{1}{x} . y^{-1} - \frac{\log x}{x} = 0.$$

Assuming, now, $y^{-1} = z$, we have

$$y^{-2} . \frac{dy}{dx} = -\frac{dz}{dx},$$

so that the given equation takes the form

$$\frac{dz}{dx} - \frac{1}{x}z + \frac{\log x}{x} = 0.$$

The general integral of this linear equation is found, Art. 68, to be

$$z = cx + \log x + 1; \qquad (c = \text{const.})$$

so that the general integral of the given equation is

$$y^{-1} = cx + \log x + 1,$$

or $$y = (cx + \log x + 1)^{-1}.$$

70. The types of differential equations which admit of given groups of one parameter, like those which have been discussed, Art. 62-68, may be multiplied indefinitely; and in this fact may be recognized a part of the extraordinary fruitfulness of the Theory of Transformation Groups as a foundation for the Theory of Differential Equations. As we know, in order to find the differential equations of the first order which are invariant under a G_1

$$Uf \equiv \xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y}.$$

it is only necessary to find the value of η' from the equation

$$\eta' \equiv \frac{d\eta}{dx}-y'\frac{d\xi}{dx},$$

and then find two independent integral-functions of the simultaneous system

$$\frac{dx}{\xi}=\frac{dy}{\eta}=\frac{dy'}{\eta'}.$$

Moreover, we have shown, Art. 56, that if an integral-function u of the ordinary differential equation in two variables

$$\frac{dx}{\xi}=\frac{dy}{\eta},$$

be known, the second integral-function v can always be found by a quadrature.

Practically, therefore, it is only necessary to choose ξ and η so that the ordinary equation

$$\frac{dx}{\xi}=\frac{dy}{\eta}$$

will have an integrable form, either in being an exact equation or in assuming one of the forms discussed, Arts. 62-69, when all differential equations of the first order which are invariant under the given G_1, and which are therefore immediately integrable, may be found by quadratures. For instance, if $\xi \equiv x f_1(xy)$ and $\eta \equiv y f_2(xy)$, the above differential equation has the form

$$f_1(xy) \,.\, x dy - f_2(xy) \,.\, y dx = 0,$$

which, by Art. 67, is integrable by a quadrature. This gives us u; and, by Art. 56, v may be found by another quadrature, so that all differential equations of the first order which are invariant under the G_1

$$Uf \equiv x f_1(xy) \frac{\partial f}{\partial x} + y f_2(xy) \frac{\partial f}{\partial y}$$

may be found by two quadratures.

This method will, in general, give rise to a new class of integrable differential equations of the first order in two variables. If desired, ξ and η might now be so chosen that the equation

$$\frac{dx}{\xi} = \frac{dy}{\eta}$$

will belong to this new class. Then, of course, two quadratures will, in general, give us another new class of integrable equations, etc.

71. It will be remembered that, Art. 55, the condition that an ordinary differential equation of the first order in two variables,

$$\Omega(x, y, y') = 0,$$

shall admit of a G_1,

$$Uf \equiv \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y},$$

is that the expression

$$U'(\Omega) \equiv \xi \frac{\partial \Omega}{\partial x} + \eta \frac{\partial \Omega}{\partial y} + \eta' \frac{\partial \Omega}{\partial y'}$$

shall be zero, either identically, or by means of $\Omega = 0$. Here, of course, $U'f$ is put for the once-extended G_1

$$U'f \equiv \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'}.$$

From this condition it is often possible to find the G_1 of which a given differential equation of the first order admits, especially whenever the form of the equation suggests the G_1.

For example, the equation

$$\Omega(x, y, y') \equiv y'^3 - 4xyy' + 8y^2 = 0$$

is homogeneous in all of its terms except one. Since, by Art. 64, all homogeneous equations of the first order admit of the G_1

$$Uf \equiv x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y},$$

we are led to suspect that the above equation will admit of a G_1 of the form

$$Uf \equiv ax\frac{\partial f}{\partial x} + by\frac{\partial f}{\partial y}. \qquad (a, b \text{ consts.})$$

The corresponding extended G_1 is

$$U'f \equiv ax\frac{\partial f}{\partial x} + by\frac{\partial f}{\partial y} + (b-a)y'\frac{\partial f}{\partial y'};$$

and the condition that the given equation $\Omega = 0$ shall be invariant under this G_1 is that $U'(\Omega)$ shall be zero identically, or by means of $\Omega = 0$. That is,

$$ax\frac{\partial}{\partial x}(y'^3 - 4xyy' + 8y^2) + by\frac{\partial}{\partial y}(y'^3 - 4xyy' + 8y^2)$$
$$+ (b-a)y'\frac{\partial}{\partial y'}(y'^3 - 4xyy' + 8y^2)$$
$$\equiv -4axyy' - 4bxyy' + 16by^2 + 3(b-a)y'^3 - 4(b-a)xyy'$$

must be zero identically, or by means of $\Omega = 0$.

Comparing this expression with the expression

$$y'^3 - 4xyy' + 8y^2,$$

we must attempt to choose b and a in such manner as to make the first expression equivalent to the second multiplied by a constant factor. It is only thus, in this case, that the condition of invariance can be satisfied, since the first expression cannot be zero identically unless $a = b = 0$, in which case Uf would vanish. If α represent any constant we see that we must have, in order that

$$3(b-a)y'^3 - \{4a + 4b + 4(b-a)\}xyy' + 16by^2 \equiv \alpha(y'^3 - 4xyy' + 8y^2),$$

the identities

$$\begin{aligned} 16b &\equiv 8a, \\ 4a+4b+4(b-a) &\equiv 4a, \\ 3(b-a) &\equiv a. \end{aligned}$$

These equations are not contradictory, and may evidently be satisfied by assuming $a=2$; $b=1$; $a=\frac{1}{3}$. Hence the given differential equation admits of the G_1:

$$Uf \equiv \frac{x}{3}\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}.$$

If the equations for determining a and b had proved to be contradictory, it would, of course, have meant that the given differential equation did not admit of a G_1 of the form

$$Uf \equiv ax\frac{\partial f}{\partial x} + by\frac{\partial f}{\partial y}.$$

In a manner similar to the above, it may be readily seen that the differential equation

$$(x^3y-3x)y'-(2x^2x^2+y)=0$$

admits of
$$Uf \equiv ax\frac{\partial f}{\partial x} + by\frac{\partial f}{\partial y},$$

when $a=1$, $b=-2$, as we should be led to expect from Art. 67. That is, the above equation admits of

$$x\frac{\partial f}{\partial x} - 2y\frac{\partial f}{\partial y}.$$

72. The types of integrable differential equations established, Arts. 51–69 illustrate the fact, mentioned Art. 17, that every ordinary differential equation of the first order has one general integral. The rigid proof of this proposition is, as already stated, Art. 17, a theorem pertaining to the Theory of Functions; but a further illustration is afforded by the fact that when the variables x and y are connected by an equation of the form

$$Xdy - Ydx = 0, \quad \text{..............................} (1)$$

we may always, by means of an infinite series, express y as a function of x and one arbitrary constant. We shall not investigate the question as to whether this series always converges or not.

From (1) we have

$$y' = \frac{Y}{X},$$

or, as we may more conveniently write it,

$$y' = f_1(x, y). \quad \text{..............................} (2)$$

By differentiating (2) we find

$$\begin{aligned} y'' &= \frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial y} \cdot y' \\ &= \frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial y} \cdot f_1. \end{aligned}$$

Let us, for brevity, write this

$$y'' = f_2(x, y). \qquad \text{..............................(3)}$$

By successive differentiations of (3) we find, analogously,

$$\begin{aligned} & y''' = f_3(x, y), \\ & \text{.................} \\ & y^{(n)} = f_n(x, y). \end{aligned}$$

Now let $\phi(x)$ be the general value of y; and when we assign a particular numerical value x_0 to x, let the corresponding value of y be designated by y_0. Here x_0 and y_0 are called the *initial values* of x and y. Then, by Taylor's Theorem, we have

$$y = \phi(x_0) + \phi'(x_0)(x - x_0) + \phi''(x_0)\frac{(x - x_0)^2}{1\,.\,2} + \dots \qquad \text{..............(4)}$$

But $\phi(x_0)$ is y_0; $\phi'(x_0)$ is the value of y' when $x = x_0$; $\phi''(x_0)$ is the value of y'' when $x = x_0$, etc. Hence, by (2), (3), etc.,

$$y = y_0 + f_1(x_0, y_0)(x - x_0) + f_2(x_0, y_0)\frac{(x - x_0)^2}{1\,.\,2} + \dots; \qquad \text{............(5)}$$

and this is an expression for the general integral of (1).

Since x_0 is a particular numerical value of x, it is seen that the general integral (5) contains only *one* arbitrary constant, y_0.

In Chapter X. we shall see that the general integral of a differential equation of the m^{th} order may be similarly expressed by an infinite series.

73. In the following examples of differential equations of the first order to be integrated, the test for an *exact* differential equation should first be applied. It will be remembered that the equation

$$X dy - Y dx = 0$$

is *exact*, if

$$\frac{\partial X}{\partial x} \equiv -\frac{\partial Y}{\partial y};$$

and the integral may be found by a quadrature in the form

$$\int Y dx - \int\left(X + \frac{\partial}{\partial y}\int Y dx\right) dy = \text{const.}$$

If the given differential equation is *not* exact, but belongs to one of the types of Arts. 62–68, it may be integrated, as already seen, by a quadrature.

In case the given differential equation does not belong to one of the types established, Arts. 62–68, the method of Art. 71 should be employed to find the G_1 of which the equation admits.

We give below a table of types of the most important of the simpler G_1 in the plane, with the corresponding type of invariant differential equation. The reader will do well to re-establish for himself those types given below which were not established, Arts. 62–68.

Group of One Parameter.	*Type of Invariant Differential Equation.*
(1) $Uf \equiv \frac{\partial f}{\partial x}$.	(1) $y' = F(y)$.
(2) $Uf \equiv \frac{\partial f}{\partial y}$.	(2) $y' = F(x)$.
(3) $Uf \equiv \frac{1}{a}\frac{\partial f}{\partial x} - \frac{1}{b}\frac{\partial f}{\partial y}$.	(3) $y' = F(ax + by)$.

It is seen that (3) includes types (1) and (2).

(4) $Uf \equiv x\frac{\partial f}{\partial x}$.	(4) $y' = \frac{F(y)}{x}$.
(5) $Uf \equiv x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$.	(5) $y' = F\left(\frac{y}{x}\right)$.

Equations of the form $(a'x + b'y + c')dy - (ax + by + c)dx = 0$ may usually be brought to the homogeneous form. See Art. 65.

(6) $Uf \equiv -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}$.	(6) $\frac{xy' - y}{x + yy'} = F(x^2 + y^2)$.
(7) $Uf \equiv x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y}$.	(7) $f_1(xy)x\,dy - f_2(xy)y\,dx = 0$.
(8) $Uf \equiv e^{\int \phi(x)dx}\cdot\frac{\partial f}{\partial y}$.	(8) $y' - \phi(x)y - \psi(x) = 0$.

The form $y' - \phi(x) . y - \psi(x) y^n = 0$ may be reduced to this one. See Art. 69.

(9) $Uf \equiv \frac{\partial f}{\partial x} + \frac{y}{x} \frac{\partial f}{\partial y}.$	(9) $y' = \frac{y}{x} + x\mathrm{F}\left(\frac{y}{x}\right).$
(10) $Uf \equiv 2x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}.$	(10) $y'y = \mathrm{F}\left(\frac{y^2}{x}\right).$
(11) $Uf \equiv x^2 \frac{\partial f}{\partial x} + xy \frac{\partial f}{\partial y}.$	(11) $\Omega\left(\frac{y}{x}, y - xy'\right) = 0.$
(12) $Uf \equiv x \frac{\partial f}{\partial x} + 2y \frac{\partial f}{\partial y}.$	(12) $xy' = \mathrm{F}\left(\frac{x^2}{y}\right).$

EXAMPLES.

(1) $(y^2 - 4xy - 2x^2)dy + (x^2 - 4xy - 2y^2)dx = 0.$

(2) $\frac{2x\,dx}{y^3} + \left(\frac{1}{y^2} - \frac{3x^2}{y^4}\right)dy = 0.$

(3) $(1 + e^{\frac{x}{y}})dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0.$

(4) $(m\,dx + n\,dy)\sin(mx + ny) = (n\,dx + m\,dy)\cos(nx + my).$

(5) $\frac{x\,dx + y\,dy}{\sqrt{1 + x^2 + y^2}} + \frac{y\,dx - x\,dy}{x^2 + y^2} = 0.$

(6) $e^x(x^2 + y^2 + 2x)dx + 2ye^x dy = 0.$

(7) $(y - x)dy + y\,dx = 0.$

(8) $(2\sqrt{xy} - x)dy + y\,dx = 0.$

(9) $x\,dy - (y + \sqrt{x^2 + y^2})dx = 0.$

(10) $(x + y)dy - (y - x)dx = 0.$

(11) $x \cos\frac{y}{x} dy - \left(y \cos\frac{y}{x} - x\right)dx = 0.$

(12) $(5y + 7x)dy + (8y + 10x)dx = 0.$

(13) $x\,dy - (y + \sqrt{x^2 - y^2})dx = 0.$

(14) $(2xy^2 - x^3)dy + (y^3 - 2yx^2)dx = 0.$

(15) $(x^4 - 2xy^3)dy + (y^4 - 2x^3y)dx = 0.$

(16) $(2y - x - 1)dy + (2x - y + 1)dx = 0.$

(17) $(7y - 3x + 3)dy + (3y - 7x + 7)dx = 0.$

(18) $(x\,dx + y\,dy)(x^2 + y^2) + x\,dy - y\,dx = 0.$

(19) $(xy + x^2)dy + y^2 dx = 0.$

(20) $(x\sqrt{x^2 + y^2} - x^2)dy + (xy - y\sqrt{x^2 + y^2})dx = 0.$

(21) $(x^2y^2+xy)y\,dx+(x^2y^2-1)x\,dy=0.$

(22) $(x^3y^3+1)(x\,dy+y\,dx)+(x^2y^2+xy)(y\,dx-x\,dy)=0.$

(23) $(y+y\sqrt{xy})dx+(x+x\sqrt{xy})\,dy=0.$

(24) $xy\{1+\cot(xy)\}(x\,dy+y\,dx)+x\,dy-y\,dx=0.$

(25) $x\,dy-(ay+x+1)dx=0.$

(26) $(1-x^2)^2dy+y\sqrt{1-x^2}dx=(x+\sqrt{1-x^2})dx.$

(27) $(1+y^2)dx=(\tan^{-1}y-x)dy.$

(28) $dy+(y-xy^3)dx=0.$

(29) $(1-x^2)dy-(xy+xy^2)dx=0.$

(30) $x\,dy+(y-y^2\log x)dx=0.$

(31) $2xy\,dy+(x-y^2)dx=0.$

(32) $(1+x^2)dy+\left(xy-\dfrac{1}{x}\right)dx=0.$

(33) $\cos x\,dy+(\sin x+y-1)dx=0.$

(34) $(1-x^2)dy-(xy+axy^2)dx=0.$

(35) $dy-2xy(x^2y^2-1)dx=0.$

(36) $(y\log x-1)y\,dx-x\,dy=0.$

(37) $\cos x\,dy+\{y^2\cos x(1-\sin x)-y\}dx=0.$

(38) $y\,dy+(y^2-\cos x)dx=0.$

(39) $x\,dy-(y+x^m)dx=0.$

The following geometrical examples lead to ordinary differential equations of the first order, which may be readily integrated by some of the foregoing methods.

For convenience of reference hereafter, it may be noted that for a plane curve referred to rectangular coordinates,

Subtangent $\equiv\dfrac{y}{y'}$; Subnormal $\equiv yy'$;

Length of the perpendicular from origin upon tangent $\equiv\dfrac{y-xy'}{\sqrt{1+y'^2}}$;

Length of the perpendicular from origin upon normal $\equiv\dfrac{x+yy'}{\sqrt{1+y'^2}}$;

Intercept of tangent upon x-axis $\equiv y-xy'$;

Intercept of tangent upon y-axis $\equiv x-\dfrac{y}{y'}$;

Distance from the origin to the foot of the normal $\equiv x+yy'$.

(40) Find the curve whose subtangent is proportional to the abscissa (k times the abscissa) of the point of contact.

(41) Find the curve whose subtangent is constant, and equal to a.

(42) Find the curve whose subnormal is constant, and equal to $2a$.

(43) Find the curve in which the angle between the *radius vector* and the tangent is one-half the vertical angle.

(44) Find the curve in which the subnormal is proportional to the n^{th} power of the abscissa.

(45) Find the curve in which the perpendicular from the origin upon the tangent is equal to the abscissa.

(46) Find a curve such that the area included between the curve, the axis of x, and an ordinate, is proportional to the ordinate.

(47) Find the curve in which the subtangent is the arithmetical mean between the abscissa and the ordinate.

(48) Find the curve in which the intercept of the normal upon the x-axis is proportional to the radius vector.

(49) Find the curve in which the intercept of the tangent upon the y-axis is proportional to the radius vector.

(50) Find the curve in which the subtangent is equal to

$$mx+ny.$$

(51) Find all differential equations of the first order which are invariant under the G_1

$$Uf \equiv y\frac{\partial f}{\partial x}.$$

(52) Find all differential equations of the first order which are invariant under the G_1

$$Uf \equiv x\frac{\partial f}{\partial y}.$$

CHAPTER V.

GEOMETRICAL APPLICATIONS OF THE INTEGRATING FACTOR. ORTHOGONAL TRAJECTORIES, AND ISOTHERMAL SYSTEMS.

74. In this chapter we propose to give some of the simpler geometrical applications of the integrating factor, Art. 59, of a differential equation of the first order in two variables.

75. In Art. 59 it was shown that if a differential equation of the form

$$X dy - Y dx = 0 \quad \text{.......................(1)}$$

admits of the G_1, which is not trivial,

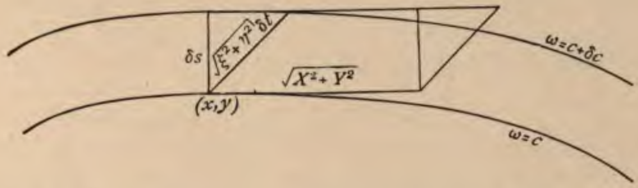
$$Uf \equiv \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y},$$

then
$$U \equiv \frac{1}{X\eta - Y\xi}$$

is an integrating factor of (1).

Suppose now that $\omega(x, y) = \text{const.}$ represents the ∞^1 integral curves of (1); then by means of Uf each curve $\omega = c$ passes over into the position of the adjoining curve $\omega = c + \delta c$. At the same time every point of general position (x, y) passes through an infinitesimal distance, Art. 29, $\sqrt{\xi^2 + \eta^2}\,\delta t$, of which the projections upon the axes of coordinates are $\xi \delta t$ and $\eta \delta t$.

Now draw the tangent to the curve $\omega=c$ at the point (x, y); and lay off upon this tangent the distance $\sqrt{X^2+Y^2}$, of which the projections upon the axes are



X and Y respectively. The two distances $\sqrt{\xi^2+\eta^2}\,\delta t$ and $\sqrt{X^2+Y^2}$ determine a parallelogram of which the area, by a proposition of Analytical Geometry, is

$$(X\eta-Y\xi)\delta t,$$

or

$$\frac{1}{U}\cdot\delta t.$$

But this parallelogram, if we neglect infinitesimals of an order higher than the first, is equal in area to the rectangle constructed upon the base $\sqrt{X^2+Y^2}$ with the altitude δs,—δs being the distance from the curve $\omega=c$ to the curve $\omega=c+\delta c$, measured at the point x, y. Hence we have

$$\frac{1}{M}\cdot\delta t=\delta s\,.\,\sqrt{X^2+Y^2},$$

or

$$M\equiv\frac{\delta t}{\sqrt{X^2+Y^2}\,.\,\delta s}.$$

Hence we see that if M is an integrating factor of a given differential equation (1), *M is inversely proportional to the area of the rectangle, one side of which is the perpendicular distance, measured at a point of general position (x, y), between the integral curve through that point and the integral curve of the family at an* *infinitesimal distance from that one; while the other side of the rectangle is the distance* $\sqrt{X^2+Y^2}$, *measured off upon the tangent to the curve through the point* (x, y), *and from that point.*

76. Let us apply the above result to a simple example.

If equal distances, of length n, are laid off upon all the normals of a given curve

$$\psi(x, y)=0,$$

the end points of the normals form a new curve. If, now, n varies, we find a family of ∞^1 curves which are called the *parallel* curves of the curve $\psi=0$. The differential equation

$$Xdy-Ydx=0 \text{ ....} (1)$$

may always be integrated by a quadrature, if its integral curves are a family of parallel curves. For in this case the perpendicular distance between two adjoining integral curves is constant: so that, by Art. 75,

$$M\equiv\frac{1}{\sqrt{X^2+Y^2}}$$

must be an integrating factor.

Hence, *if it is known that the differential equation* (1) *represents a family of* ∞^1 *parallel curves,*

$$M\equiv\frac{1}{\sqrt{X^2+Y^2}}$$

is an integrating factor of (1).

Conversely, it is easy to see that if

$$M\equiv\frac{1}{\sqrt{X^2+Y^2}}$$

be an integrating factor of (1), the distance between two adjoining curves must be *constant*, and the curves are parallel.

The ∞^1 *involutes* of a given curve form a family of parallel curves; and hence their differential equation may always be integrated by a quadrature.

For example the involutes of the parabola

$$y = x^2$$

are represented, as is shown in the Differential Calculus, by the equation

$$2(x+\sqrt{x^2-y})dy+dx=0.$$

Hence
$$M \equiv \frac{1}{\sqrt{4(x+\sqrt{x^2-y})^2+1}}$$

must be an integrating factor of the above equation, as may be at once verified.

77. An *orthogonal trajectory* is a curve which intersects at right angles each member of a given family of ∞^1 curves.

A family of ∞^1 curves, represented by a differential equation

$$Xy' - Y = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

will evidently have ∞^1 orthogonal trajectories; and their differential equation is readily obtained from (1). For, at any point of general position (x, y), the integral curve of (1) through that point is perpendicular to the orthogonal trajectory through the point; hence, if y' be the tangential direction of the integral curve, $-\frac{1}{y'}$ must be that of the orthogonal trajectory. Thus if we substitute in (1) $-\frac{1}{y'}$ for y', we obtain the differential equation of the ∞^1 orthogonal trajectories of the integral curves of (1), in the form

$$X + Yy' = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Reciprocally, the integral curves of (1) are the orthogonal trajectories of (2).

Example. It is required to find the orthogonal trajectories of the hyperbolas

$$xy=a. \qquad (a=\text{parameter})$$

The differential equation of these ∞^1 curves is obviously

$$xdy+ydx=0,$$

or

$$xy'+y=0.$$

Writing $-\frac{1}{y'}$ in place of y', we find as the differential equation of the orthogonal trajectories

$$x-yy'=0,$$

or

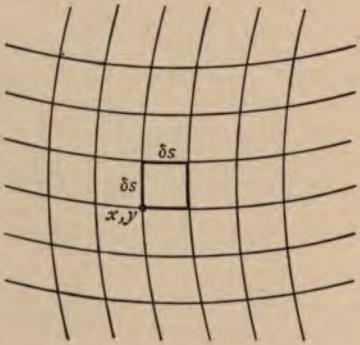
$$xdx-ydy=0.$$

The variables are here separate, so that we find at once the integral curves

$$x^2-y^2=c; \qquad (c=\text{parameter})$$

which is also a family of hyperbolas.

78. A family of ∞^1 curves in the plane is said to be *isothermal* when, together with their orthogonal trajectories, they form a network of infinitesimal squares.



If

$$Xdy-Ydx=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

represent a family of isothermal curves, their orthogonal trajectories will, Art. 77, of course, be represented by

$$Xdx+Ydy=0,\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

and both of the equations (1) and (2) may be integrated by quadratures by means of the geometrical interpretation of the integrating factor, Art. 75.

For, if we consider any two adjoining curves of each of the families (1) and (2) which form a small square at the point (x, y), the breadth, δs, of the strips enclosed by both pairs of curves is the same, since δs is the side of the infinitesimal square.

Hence
$$M \equiv \frac{\delta t}{\sqrt{X^2+Y^2}\,\delta s}$$
is an integrating factor of both (2) and (1).

But if two ordinary differential equations of the forms (2) and (1) have a *common* integrating factor, this factor may be determined by a quadrature. For, if M be the common integrating factor, by Art. 52, M must satisfy the equations
$$\frac{\partial MX}{\partial x}+\frac{\partial MY}{\partial y}=0,$$
$$\frac{\partial MY}{\partial x}-\frac{\partial MX}{\partial y}=0;$$
or,
$$X\frac{\partial \log M}{\partial x}+Y\frac{\partial \log M}{\partial y}=-\frac{\partial X}{\partial x}-\frac{\partial Y}{\partial y},$$
$$Y\frac{\partial \log M}{\partial x}-X\frac{\partial \log M}{\partial y}=-\frac{\partial Y}{\partial x}+\frac{\partial X}{\partial y}. \quad\text{..........(3)}$$

From the last two equations $\frac{\partial \log M}{\partial x}$ and $\frac{\partial \log M}{\partial y}$ may be determined as functions of x and y; and, if these quantities satisfy the condition of integrability, Art. 51,
$$\frac{\partial}{\partial y}\frac{\partial \log M}{\partial x}\equiv\frac{\partial}{\partial x}\frac{\partial \log M}{\partial y}, \quad\text{..................(4)}$$
we may find $\log M$, or M itself, by a quadrature from the exact equation
$$d\log M\equiv\frac{\partial \log M}{\partial x}dx+\frac{\partial \log M}{\partial y}dy=0.$$

If the above condition of integrability were not satisfied, (1) and (2) would have no common integrating factor; and hence the families of integral curves would not be isothermal. Thus, that (3) shall give such values for $\frac{\partial \log M}{\partial x}$ and $\frac{\partial \log M}{\partial y}$ as satisfy (4) is a necessary condition that (1) shall represent an isothermal family. This condition is also sufficient; for if (4) is satisfied, (1) and (2) have a common integrating factor—that is, the quantity designated as δs above must be the same for both families of integral curves, and these integral curves form a net-work of small squares, or are isothermal.

If (1) represents an isothermal family of curves, therefore, the common integrating factor M, of (1) and (2), may be found by a quadrature; and the equations (1) and (2) may be integrated by another quadrature each.

Example 1. The differential equation

$$2xy\,dy-(y^2-x^2)dx=0, \qquad \text{..........................(5)}$$

represents an isothermal family of curves. For the orthogonal trajectories are represented by

$$(y^2-x^2)dy+2xy\,dx=0; \qquad \text{..........................(6)}$$

so that the equations (3) have the forms

$$2xy\frac{\partial \log M}{\partial x}+(y^2-x^2)\frac{\partial \log M}{\partial y}=-4y;$$

$$(y^2-x^2)\frac{\partial \log M}{\partial x}-2xy\frac{\partial \log M}{\partial y}=4x.$$

Hence $$\frac{\partial \log M}{\partial x}=\frac{-4x}{x^2+y^2}, \quad \frac{\partial \log M}{\partial y}=\frac{-4y}{x^2+y^2},$$

and it may be immediately verified that the condition (4) is satisfied. Thus (5) represents an isothermal family; and the integrating factor of (5) and (6) is obtained from

$$d\log M\equiv-2\left\{\frac{2x}{x^2+y^2}dx+\frac{2y}{x^2+y^2}dy\right\}=0$$

by a quadrature. We find

$$\log M \equiv -2\log(x^2+y^2);$$

and hence

$$M \equiv \frac{1}{(x^2+y^2)^2}.$$

It may be at once verified that this is an integrating factor of (5) and also of (6). Hence, from (5) and (6) respectively, we find by quadratures

$$\frac{x}{x^2+y^2} = \text{const.}, \quad \frac{y}{x^2+y^2} = \text{const.}$$

as integrals.

The first isothermal family is that of all circles which touch the y-axis at the origin; the second is that of all circles which touch the x-axis at the origin.

Of course equations (5) and (6) might also have been integrated by the method of Art. 64.

79. In the following examples, such differential equations as represent curve-families consisting either of isothermal or of parallel curves may be integrated by the method of Arts. 76–78. It may usually be seen, from the geometrical meaning of the equation given, whether the orthogonal family will be isothermal or not.

Those differential equations which represent orthogonal trajectories which are neither parallel nor isothermal curves may be integrated by Art. 73.

EXAMPLES.

Find the orthogonal trajectories of the following curve-families:

(1) The straight lines $y = ax$. ($a \equiv$ parameter.)

(2) The parabolas $y^2 = 4ax$.

(3) The circles $x^2+y^2=a^2$.

(4) The parabolas $y^2 = 4a(x+a)$.

(5) The ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. ($a \equiv$ const., $b \equiv$ parameter.)

(6) The circles $x^2-2ax+y^2-2ay+a^2=0$.

(7) The ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=n^2$. ($n \equiv$ parameter.)

(8) The circles $x^2+y^2+ax-1=0$

(9) The confocal conics $\frac{x^2}{b^2+c^2}+\frac{y^2}{b^2}=1.$ ($b\equiv$ parameter.)

(10) The parallel curves $y-a=\phi(x)$,
which result from translating the curve $y=\phi(x)$ parallel to itself along the y-axis.

(11) Apply the result of Ex. (10) to the case of the semi-cubical parabola

$$y=\frac{2}{3}\sqrt{\frac{x^3}{n}}.$$

(12) Show that the differential equation of the orthogonal trajectories of the curves in polar coordinates

$$F(\rho,\ \theta,\ c)=0$$

is obtained by eliminating c between the above equation and

$$\frac{\partial F}{\partial \theta}\cdot\frac{d\rho}{d\theta}-\rho^2\frac{\partial F}{\partial \rho}=0.$$

(13) Find the orthogonal trajectories of the curves

$$\rho=\log\tan\theta+a.$$

(14) Find the orthogonal trajectories of the curves

$$\theta-a=\phi(\rho),$$

which result from rotating the curve

$$\theta=\phi(\rho)$$

around a fixed point in the plane.

(15) Apply the result of Ex. (11) to find the orthogonal trajectories of the circles which result from rotating the circle

$$\rho=b\cos\theta.$$

around one end of its diameter.

CHAPTER VI.

DIFFERENTIAL EQUATIONS OF THE FIRST ORDER, BUT NOT OF THE FIRST DEGREE. SINGULAR SOLUTIONS.

80. We propose to give in Sec. I. of this Chapter methods for integrating some of the simpler differential equations in two variables which are of the first order, but not of the first degree.

In Sec. II. we shall see that a differential equation of a degree higher than the first is sometimes satisfied by a function which is not a function of the "integral-function" of the equation. This peculiar function, equated to zero, constitutes what is known as a "Singular Solution" of the given equation. A simple method for finding the Singular Solution—when one exists—of an invariant differential equation of the first order will be given.

SECTION I.

Differential Equations of a Degree Higher than the First.

81. In Art. 55 it was shown that the condition that the differential equation of the first order

$$\Omega(x, y, y')=0$$

shall admit of a given G_1

$$U'f \equiv \xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y}+\eta'\frac{\partial f}{\partial y'}$$

is that the expression $U'(\Omega)$ shall be zero, either identically, or by means of $\Omega=0$.

It is clear that this condition is independent of the form of Ω, and hence of the degree of the equation $\Omega=0$. It will be borne in mind, however, that in order to apply the method of Art. 59 to integrate the equation $\Omega=0$ by a quadrature, when it is known that this equation admits of a given G_1, $U'f$, it is necessary to solve $\Omega=0$ in terms of y', in the form,

$$Xy'-Y=0.$$

The algebraic solution of $\Omega=0$ in terms of y' is not always simple; and in Cases II. and III. below we shall indicate methods by means of which that work may sometimes be simplified or avoided.

82. *Case I.* Suppose that the equation

$$\Omega(x, y, y')=0 \text{ ....(1)}$$

may be solved, algebraically, in terms of y' in such manner that the resulting roots will be *rational* functions of x and y. Thus, if (1) is of the n^{th} degree, it may be written

$$(y'-\phi_1(x, y))(y'-\phi_2(x, y)) \dots (y'-\phi_n(x, y))=0, \text{ ...(2)}$$

where the $\phi_1, \dots, \phi_n$ are *rational* functions of x and y.

In this case, since (1) can be resolved into the linear factors in (2), (1) is called a "decomposable" equation.

But (2) is satisfied by writing

$$y'-\phi_1(x, y)=0, \quad y'-\phi_2(x, y)=0, \dots \, y'-\phi_n(x, y)=0; \text{...(3)}$$

and if the general integrals of the n equations (3) have been found, by the methods of Chap. IV., in the form

$$y-\Phi_1(x, y, c_1)=0, \; y-\Phi_2(x, y, c_2)=0, \dots \, y-\Phi_n(x, y, c_n)=0,$$

the general integral of (1) will have the form

$$\{y-\Phi_1(x,y,c_1)\}\{y-\Phi_2(x,y,c_2)\},\dots\{y-\Phi_n(x,y,c_n)\}=0. \quad (4)$$

But equation (4) loses nothing of its generality if we assume the arbitrary constants all equal to *one* constant, say c. For, in order to find any value of y, it is necessary to equate to zero one of the n factors on the left-hand side of (4), which gives an equation of the form

$$y-\Phi_k(x,\ y,\ c)=0. \quad \dots\dots\dots\dots(5)$$

Now since c is an arbitrary constant, by giving c all possible values, the form (5) may be made to contain *all* the integrals which may be derived from the corresponding k^{th} factor of (2).

Hence we find as the general integral of the original decomposable differential equation (1):

$$\{y-\Phi_1(x,\ y,\ c)\}\dots\{y-\Phi_n(x,\ y,\ c)\}=0.$$

Example. Suppose (2) to be of the second degree, of the form

$$y'^2-(x+y)y'+xy=0. \quad \dots\dots\dots\dots(6)$$

This equation may be written

$$(y'-x)(y'-y)=0,$$

whence $$y'-x=0,\ \ y'-y=0.$$

Integrating the last two equations we find

$$y-\frac{x^2+c}{2}=0,\ \ y-ce^x=0.$$

so that the general integral of (6) is

$$\left(y-\frac{x^2+c}{2}\right)(y-ce^x)=0.$$

83. *Case II.* If the given equation

$$\Omega(x,\ y,\ y')=0$$

can be readily solved with respect to y, in the form

$$y=\phi(x,\ y'), \quad \dots\dots\dots\dots(7)$$

it is sometimes best to differentiate (7), regarding y' as a variable as well as x and y, and substituting in the result $y'dx$ for dy. A differential equation of the first degree between x and y' must result. Integrate this equation by the methods of Chapter IV., and eliminate y' between its primitive and the equation $\Omega=0$.

Example. Given $\quad y=xy'^2+2y'.$

Hence $$dy=y'^2dx+2xy'dy'+2dy',$$

or, putting $y'dx$ for dy,

$$(y'^2-y')dx+2(xy'+1)dy'=0.$$

This is an example of Art. 68; and we find as an integrating factor

$$e^{\int\frac{1}{y'(y'-1)}dy'}=\frac{y'-1}{y'}.$$

Thus we have to integrate the exact differential equation

$$(y'-1)^2dx+\frac{2(xy'+1)(y'-1)}{y'}dy'=0.$$

By Art. 48, we find as the general integral of this equation,

$$(y'-1)^2x-2(\log y'-y')=\text{const.}$$

The general integral of the first equation is to be found by eliminating y' between that equation and the last one.

84. *Case III.* When the differential equation $\Omega=0$ can be readily solved in terms of x, in the form,

$$x-\phi(y,y')=0, \text{........................}(8)$$

it is sometimes best to differentiate (8), regarding y' as variable as well as x and y, and substituting $\frac{dy}{y'}$ for dx. A differential equation of the first degree in terms of y and y' must result. Integrate this equation by the methods of Chapter IV., and eliminate y' between the resulting equation and $\Omega=0$.

Example. Given $yy'^2+2xy'=y.$

Hence
$$x=y\frac{1-y'^2}{2y'},$$

and
$$dx=\frac{1-y'^2}{2y'}dy-y\frac{1+y'^2}{2y'^2}dy'.$$

Putting for dx, $\frac{dy}{y'}$, we find at once,
$$\frac{dy}{y}+\frac{dy'}{y'}=0.$$

The general integral of this equation is $yy'=c$. Eliminating y' from the first equation, the general integral required is found to be
$$y^2=2cx+c^2.$$

85. An important equation of the form
$$y=x\psi(y')+\phi(y'),$$
which is known as Clairaut's equation, and which may be integrated in a manner analogous to that employed in Case II., will be treated separately in the next Chapter.

SECTION II.

Singular Solutions.

86. It will be remembered that in Art. 59 it was shown that if a given differential equation of the first order
$$\Omega(x, y, y')=0, \qquad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$
admits of a known G_1
$$Uf\equiv\xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y},$$
the differential equation may always be integrated by a quadrature, *provided that the infinitesimal transformation Uf is not trivial with regard to* $\Omega=0$: that is, provided that the path-curves of Uf do not coincide with the integral curves of $\Omega=0$.

But now the question arises, may not a limited number of path-curves of the G_1 coincide with particular integral curves of $\Omega=0$, when the G_1 is *not* trivial?

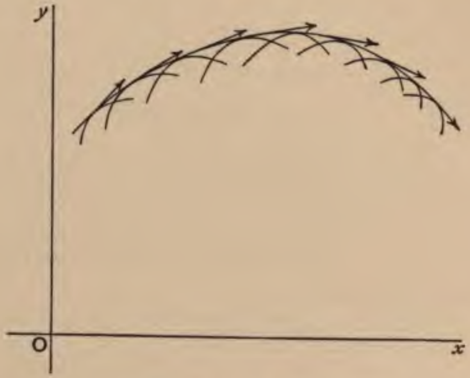
It will be found that such may be the case. For, along curves which are at once path-curves of the G_1 and integral curves of $\Omega=0$, the value of y' given by the G_1 must coincide with the value of y' given by $\Omega=0$. Hence, to find such curves we only need to substitute

$$y'=\frac{\eta}{\xi}$$

in $\Omega=0$, and the resulting equation

$$\Omega\left(x,\ y,\ \frac{\eta}{\xi}\right)=0, \qquad\qquad (2)$$

will give the path-curves of the G_1 which are also integral curves of (1), *if such exist.*



But it is easy to see that we may also find in this way the equation to a curve which is a path-curve of the G_1 and which satisfies the given differential equation (1), but which is *not* a particular integral-curve of (1).

For it may happen that the family of integral-curves (1) have an *envelope*, and if so, it is clear that the equation to the envelope will satisfy the differential equation (1); for at any point on the envelope the direction of the tangent to the envelope is the same as that of the tangent to either of the two consecutive curves of the family, which, from the Differential Calculus, we know must coincide in that point. Hence the value of y' at any point on the envelope will satisfy the differential equation; and the equation to the envelope is called a "Singular Solution" of (1). Since at any point on the envelope two values of y' given by (1) must coincide, it is clear that equation (1) must be of at least the second degree in y' in order that the integral-curves may have an envelope.

But now the family of integral-curves of (1) is invariant under the transformation Uf; hence it is clear that the envelope of the family, if one exists, is an invariant curve of which the points are interchanged by means of the transformation Uf. In other words, the envelope must be a *path-curve* of the G_1, Uf, of which (1) admits. To find this particular path-curve, we only need to substitute $\frac{\eta}{\xi}$ for y' in (1); and the resulting curve, or curves, must, as we saw, be those curves in the plane for which the values of y' given by the differential equation (1), and by the G_1, are the same. Hence, we find by this method the singular solution of (1), if one exists; and, occasionally, as indicated above, a limited number of particular integral-curves of the differential equation, which are, at the same time, path-curves of the G_1.

The particular integral-curves may be distinguished from the singular solution by the fact that the equation to a particular integral-curve may always be obtained from the general integral of (1) by assigning a special value to the constant of integration, while the equation to the singular solution cannot be so obtained.

If the equation (2) breaks up into factors, each factor must be separately examined to see whether it is a particular integral or a singular solution.

It may be remarked that ξ and η cannot both be zero along the enveloping curve of an invariant family; for, as we saw above, the points of the envelope are interchanged when the curves of the invariant family are interchanged, whereas all points on curves along which $\xi \equiv \eta \equiv 0$, are absolutely invariant.

Example 1. Given

$$y'^2y^2\cos^2\alpha - 2y'xy\sin^2\alpha + y^2 - x^2\sin^2\alpha = 0.$$

This equation is homogeneous, and hence is invariant under the G_1,

$$Uf \equiv x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}.$$

Thus, according to the above theory, we find the singular solution, if one exists, by substituting in the above equation $\frac{y}{x}$ for y'. We obtain, after an obvious reduction, the two equations

$$x^2 + y^2 = 0,$$

$$x^2 = (x^2 + y^2)\cos^2\alpha.$$

The general integral is found by Art. 59 to be

$$x^2 + y^2 - 2cx + c^2\cos^2\alpha = 0; \qquad (c = \text{const.})$$

and hence we see that

$$x^2 + y^2 = 0,$$

which may be obtained from the general integral by assigning to the arbitrary constant c the value zero, is a particular integral; while

$$x^2 = (x^2 + y^2)\cos^2\alpha,$$

which may be written

$$y = \pm x\tan\alpha$$

must constitute a singular solution, since these equations satisfy the given differential equation, and cannot be obtained by assigning any special value to c in the general integral.

Example 2. We know, Art. 71, that the differential equation

$$y'^3-4xyy'+8y^2=0$$

admits of the G_1,

$$Uf\equiv\frac{x}{3}\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}.$$

To find the singular solution, if one exists, we substitute for y', in the above equation, $\frac{3y}{x}$. Hence

$$27y^3-4x^3y^2=0,$$

or

$$y=0,\quad y-\frac{4x^3}{27}=0.$$

The general integral is

$$y=c_1(x-c)^2;$$

and hence we see that $y=0$ is a particular integral, while

$$y-\frac{4x^3}{27}=0$$

is the singular solution.

Example 3. The differential equation

$$y'^2(1-x^2)-x^2=0,$$

being free of y, admits of the G_1

$$Uf\equiv\frac{\partial f}{\partial y}.$$

Since for this G_1, $\delta y=1$, $\delta x=0$, we have

$$\frac{\delta y}{\delta x}\equiv y'\equiv\infty,\quad \frac{1}{y'}\equiv 0.$$

Therefore we write the above equation

$$(1-x^2)-\frac{x^2}{y'^2}=0;$$

and, substituting for y' the value ∞, we find

$$x=\pm 1.$$

This is a singular solution, since the differential equation possesses the general integral

$$x^2+(y-a)^2=1.$$

The geometrical meaning of the singular solution in connection with the ∞^1 curves represented by the general integral is obvious.

EXAMPLES.

Integrate the following differential equations, finding the singular solutions, when such exist, as well as the general integrals. For types of invariant equations, see Art. 73.

(1) $y'^2-5y'+6=0.$
(2) $y'^2-a^2y^2=0.$
(3) $x^2y'^2+3xyy'+2y^2=0.$
(4) $y'(y'+y)=x(x+y).$
(5) $y'^3+2xy'^2-y^2y'^2-2xy^2y'=0.$
(6) $y'^2+2yy'\cot x=y^2.$
(7) $y=(1+x)y'^2.$
(8) $yy'^2+2xy'-y=0.$
(9) $3xy'^2-6yy'+x+2y=0.$
(10) $3y^2y'^2-2xyy'+4y^2-x^2=0.$
(11) $y=-xy'+x^4y'^2.$
(12) $xy'^2-2yy'+ax=0.$
(13) $y=ay'+by'^2.$ (Art. 83.)
(14) $x^2+y=y'^2.$
(15) $y^2=x^2(1+y'^2).$
(16) $y=y'^2+2y'^3.$
(17) $y'^2y+2xy'=y.$ (Art. 84.)
(18) $x=y'+\log y'.$
(19) $x^2y'^2=1+y'^2.$
(20) $my-nxy'=yy'^2.$
(21) $xy^2(y'^2+2)=2y'y^3+x^3.$
(22) $yy'^2+(x-y)y'=x.$
(23) $xy'^2-2yy'+4x=0.$
(24) $y'^4=4y(xy'-2y)^2.$
(25) $4y'^2x(x-a)(x-b)=\{3x^2-2x(a+b)+ab\}^2.$
(26) $a^2yy'^2-4xy'+y=0.$
(27) $y'^2+2x^3y'=4x^2y.$

CHAPTER VII.

RICCATI'S EQUATION AND CLAIRAUT'S EQUATION.

87. We propose in this Chapter to make brief mention of two important historical differential equations of the first order, which are known respectively as Riccati's equation and Clairaut's equation. The treatment of these equations sketched here will be the same as that of the ordinary text-books: for, although both equations may be treated most advantageously from the standpoint of the Theory of Transformation Groups, that method would require a more extensive knowledge of these groups than it is advisable to give in an elementary text-book.

SECTION I.

Riccati's Equation.

88. This equation takes its name from that of an Italian mathematician, Riccati, who was the first to discuss it.

The general form of Riccati's equation is

$$\frac{dy}{dx}-\phi(x)\,.\,y^2-\psi(x)\,.\,y-\chi(x)=0\,;\;\text{..........}(1)$$

but this equation can only be integrated in a few special cases: and the particular form usually discussed is

$$x\frac{dy}{dx}-ay+by^2=cx^n,\;\text{..................}(2)$$

where a, b, c, n, are certain constants. By introducing into (2) the new variables $z=x^a$, $u=\frac{y}{x^a}$, that equation takes the form

$$\frac{du}{dz}+\frac{b}{a}u^2=\frac{c}{a}z^{\frac{n}{a}-2}. \qquad \text{.....................(3)}$$

a special form of (1), which is itself sometimes designated as Riccati's equation, instead of the more general equation (1).

The equation (2) happens to be much more easy to discuss than equation (3); and it is easy to deduce from the condition that (2) shall be integrable the condition that (3) shall also be integrable. We shall first show that equation (2) is always integrable when $n=2a$; then we shall show that the integration of (2) may always be made to depend upon this case when $\frac{n\pm 2a}{2n}$ is a positive integer.

89. *Case I. The equation*

$$x\frac{dy}{dx}-ay+by^2=cx^n \qquad \text{..................(2)}$$

is always integrable when $n=2a$.

Let us assume $y=x^a v$; then (2) becomes

$$x^{1-a}\frac{dv}{dx}+bv^2=cx^{n-2a};$$

and if $n=2a$ this equation becomes

$$x^{1-a}\frac{dv}{dx}+bv^2=c,$$

or

$$\frac{dv}{c-bv^2}=\frac{dx}{x^{1-a}}. \qquad \text{.........................(3)}$$

In this equation the variables are separate, so that it may be integrated by a quadrature. If we return to the original variables, we find the exact differential equation

$$\frac{x^a dy - ayx^{a-1}dx}{by^2 - cx^{2a}} + x^{a-1}dx = 0, \ldots\ldots\ldots\ldots(4)$$

of which the general integral is given by

$$y = \left(\frac{c}{b}\right)^{\frac{1}{2}} x^a \frac{Ce^{\frac{2(bc)^{\frac{1}{2}}x^a}{a}} + 1}{Ce^{\frac{2(bc)^{\frac{1}{2}}x^a}{a}} - 1},$$

or $$y = \left(-\frac{c}{b}\right)^{\frac{1}{2}} . x^a \tan\left\{C - \frac{(-bc)^{\frac{1}{2}}x^a}{a}\right\},$$

according as b and c have the same or opposite signs—C being the arbitrary constant of integration.

90. *Case II. The equation*

$$x\frac{dy}{dx} - ay + by^2 = cx^n \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

is always integrable when $\dfrac{n \pm 2a}{2n}$ *is a positive integer.*

Let us assume

$$y = A + \frac{x^n}{y_1},$$

where A is a constant to be determined. The equation (2) is easily seen to take the form

$$-aA + bA^2 + (n - a + 2bA)\frac{x^n}{y_1} + b\frac{x^{2n}}{y_1^2} - \frac{x^{n+1}}{y_1^2} \cdot \frac{dy_1}{dx} = cx^n. \ldots(5)$$

We shall choose A so that the constant in this equation shall be zero; thus we may choose $A = \frac{a}{b}$, or $A = 0$, so that there are two subdivisions for this case of the problem.

(1) If, in the first place, $A=\frac{a}{b}$, the last equation, after a slight reduction, takes the form,

$$x\frac{dy_1}{dx}-(a+n)y_1+cy_1^2=bx^n.\dots\dots\dots\dots\dots(6)$$

It is seen that (6) is of the same form as (2), except that b and c have changed places, and a has been changed to $a+n$: and this change was brought about by substituting in (2)

$$y\equiv\frac{a}{b}+\frac{x^n}{y_1}$$

in place of y.

Hence, if in (6) we make the substitution

$$y_1\equiv\frac{a+n}{c}+\frac{x^n}{y_2},$$

it is clear that (6) will take the form

$$x\frac{dy_2}{dx}-(a+2n)y_2+by_2^2=cx^n,\dots\dots\dots\dots\dots(7)$$

where b and c have again changed places, and $a+n$ has become $a+2n$.

Thus, if λ successive substitutions of the above forms are made in the equation (2), that equation will take either the form

$$x\frac{dy_\lambda}{dx}-(a+\lambda n)y_\lambda+cy_\lambda^2=bx^n,\dots\dots\dots\dots\dots(8)$$

or the form

$$x\frac{dy_\lambda}{dx}-(a+\lambda n)y_\lambda+by_\lambda^2=cx^n,\dots\dots\dots\dots\dots(9)$$

according as λ is odd or even.

But by Case I., the equations (8) and (9) are integrable if

$$n=2(a+\lambda n),$$

that is, if $$\frac{n-2a}{2n}=\lambda.$$

(2) Secondly, let us assume $A=0$. Then (2), by means of the substitution

$$y=\frac{x^n}{y_1},$$

is readily seen to take the form

$$x\frac{dy_1}{dx}-(n-a)y_1+cy_1{}^2=bx^n,\ldots\ldots\ldots\ldots\ldots(10)$$

an equation which is identical with (6) except that a has become $-a$. If the preceding series of substitutions are now made, the final result will be found to be the same except for the sign of a. Thus the equation (2) will be integrable when

$$\frac{n+2a}{2n}=\lambda.$$

Combining these results we see that *the equation*

$$x\frac{dy}{dx}-ay+by^2=cx^n$$

is integrable whenever $\frac{n\pm 2a}{2n}$ *is a positive integer.*

From the nature of the substitutions employed in the above two cases, it is clear that the general integral of Riccati's equation, when that equation is integrable by the above method, will be given in the form of a finite continued fraction, the last denominator of which is to be found by a quadrature.

91. We found, in the last article, as a condition of integrability, that $\frac{n\pm 2a}{2n}$ should be a positive integer, say λ.

If we assume $\frac{n-2a}{2n}=\lambda$, from (1) Art. 90, we have the series of substitutions

$$y=\frac{a}{b}+\frac{x^n}{y_1},$$
$$y_1=\frac{a+n}{c}+\frac{x^n}{y_2},$$
$$y_2=\frac{a+2n}{b}+\frac{x^n}{y_3},$$

....

$$y_{\lambda-1}=\frac{a+(\lambda-1)n}{\mu}+\frac{x^n}{y_\lambda},$$

where μ has the value b or c, according as λ is odd or even.

From these equations we have

$$y=\frac{a}{b}+\cfrac{x^n}{\cfrac{a+n}{c}+\cfrac{x^n}{\cfrac{a+2n}{b}+\dots}}, \text{ ....(11)}$$

the last denominator of the finite continued fraction being

$$\frac{a+(\lambda-1)n}{\mu}+\frac{x^n}{y_\lambda}.$$

The value of y_λ is to be determined by a quadrature from one of the equations (8) or (9). These equations may now be put into exact forms, analogous to (4), writing $a+\lambda n$ for a:

$$\frac{x^{a+\lambda n}dy_\lambda-(a+\lambda n)y_\lambda x^{a+\lambda n-1}dx}{cy_\lambda^2-bx^n}+x^{a+\lambda n-1}dx=0, \text{ ...(12)}$$

and

$$\frac{x^{a+\lambda n}dy_\lambda-(a+\lambda n)y_\lambda x^{a+\lambda n-1}dx}{by_\lambda^2-cx^n}+x^{a+\lambda n-1}dx=0, \text{ ...(13)}$$

λ being supposed *odd* in (12) and *even* in (13).

If now we assume

$$\frac{n+2a}{2n}=\lambda,$$

it is easy to see that y will have the value

$$y=\cfrac{x^n}{\cfrac{n-a}{c}+\cfrac{x^n}{\cfrac{2n-a}{b}+\cfrac{x^n}{\cfrac{3n-a}{c}+\ldots}}}, \qquad \text{.........(14)}$$

where the last denominator is

$$\frac{(\lambda-1)n-a}{\mu}+\frac{x^n}{y_\lambda}.$$

Also, y_λ is to be found from one of the following exact equations, which result from (12) and (13) by changing a into $-a$:

$$\frac{x^{\lambda n-a}dy_\lambda-(\lambda n-a)y_\lambda x^{\lambda n-a-1}dx}{cy_\lambda{}^2-bx^n}+x^{\lambda n-a-1}dx=0, \ldots(15)$$

$$\frac{x^{\lambda n-a}dy_\lambda-(\lambda n-a)y_\lambda x^{\lambda n-a-1}dx}{by_\lambda{}^2-cx^n}+x^{\lambda n-a-1}dx=0, \ldots(16)$$

λ being *odd* in (15) and *even* in (16).

Thus, when $\frac{n-2a}{2n}$ is a positive integer, (11) in connection with either (12) or (13), according as λ is odd or even, will represent the required general integral.

When $\frac{n+2a}{2a}$ is a positive integer, (14) in connection with either (15) or (16), according as λ is odd or even, will represent the required general integral.

92. By making use of new constants, the equation (3) which is itself sometimes designated as Riccati's equation, may obviously be written

$$\frac{du}{dz}+bu^2=cz^m. \qquad \text{......................(17)}$$

If, in place of z and u we introduce the new independent variables x, and $y \equiv ux$, (17) becomes

$$x\frac{dy}{dx}-y+by^2=cx^{m+2}. \quad\text{..................}(18)$$

But we know that (18) is integrable when

$$\frac{(m+2)\pm 2}{2(m+2)}=\lambda,$$

where λ is a positive integer. This is therefore the condition that the special form (17), of Riccati's equation, shall be integrable.

Making use of the negative and of the positive signs in succession in the above condition, we find

$$m=\frac{-4\lambda}{2\lambda-1}, \quad\text{........................}(19)$$

and

$$m=\frac{-4(\lambda-1)}{2\lambda-1}. \quad\text{..................}(20)$$

By changing, in (20), the integer λ into $\lambda+1$, which is obviously allowable on condition that λ may assume the value zero, as well as any positive integral value, (20) may be written

$$m=\frac{-4\lambda}{2\lambda+1}. \quad\text{........................}(21)$$

Here, if $\lambda=0$, $m=0$; and since for $\lambda=0$ in (19) we also have $m=0$, it is clear that λ may admit of the same series of values in (19) as in (21).

Combining these results, we see that *Riccati's equation, in the special form* (17), *is integrable whenever*

$$m=\frac{-4\lambda}{2\lambda\pm 1},$$

λ being zero, or some positive integer.

When the negative sign is used in the last equation, the general integral is given by (11) in connection with (12) or (13) according as λ is odd or even. When the positive sign is used, the general integral is given by (14) in connection with (15) or (16) according as λ is odd or even.

Example. Given the equation

$$\frac{du}{dx} - u^2 = 2x^{-\frac{8}{3}} \text{.................................}(22)$$

This is a case of equation (17). By substituting y/x for u, the equation takes the form of (18),

$$x\frac{dy}{dx} - y - y^2 = 2x^{-\frac{2}{3}} \text{.............................}(23)$$

The condition of integrability (19)

$$m = \frac{-4\lambda}{2\lambda - 1}, \text{....................................}(19)$$

gives
$$-\frac{8}{3} = \frac{-4\lambda}{2\lambda - 1},$$

or
$$\lambda = 2.$$

Hence, the integral of (23) is given by the equation (11) in connection with (13). Here we have

$$a = 1,\ \lambda = 2,\ n = -\tfrac{2}{3},\ a + n\lambda = -\tfrac{1}{3},\ b = -1,\ c = 2\,;$$

hence (13) becomes

$$\frac{x^{-\frac{1}{3}}dy_2 + \frac{1}{3}y_2x^{-\frac{4}{3}}dx}{-y_2{}^2 - 2x^{-\frac{2}{3}}} + x^{-\frac{4}{3}}dx = 0.$$

This is an equation of the form (4), where the a, b, and c of that equation have the respective values

$$-\tfrac{1}{3},\ -1,\ 2.$$

Thus, since b and c have opposite signs, the integral of the last equation is

$$y_2 = \sqrt{2}x^{-\frac{1}{3}}\tan(C + 3\sqrt{2}x^{-\frac{1}{3}}).$$

This value of y_2 substituted in

$$y = \frac{a}{b} + \frac{x^n}{\dfrac{a+n}{c} + \dfrac{x^n}{y_2}} \text{ ....}(11')$$

gives the general integral of (23); and if in that result we restore to y its original value ux, we find the general integral of (22).

SECTION II.

Clairaut's Equation.

93. The equation of the form

$$y=xy'+\phi(y'), \text{ .... }(1)$$

is generally known as Clairaut's equation. Although an equation of the first order, it is not usually of the first degree.

Differentiating (1), regarding y' as a variable, as well as x and y, we find

$$y'=y'+\{x+\phi'(y')\}\frac{dy'}{dx}; \text{ .... }(2)$$

from which follows either

$$\frac{dy'}{dx}=0,$$

or

$$x+\phi'(y')=0. \text{ .... }(3)$$

From the former of these equations follows

$$y'=c, \qquad (c=\text{const.})$$

so that, from (1), the general integral must have the form

$$y=cx+\phi(c); \text{ .... }(4)$$

and that this is correct may be immediately verified by differentiation.

But now, if we eliminate y' between (3) and (1) we obtain a relation between x and y which satisfies (1) also, and which is therefore a solution of the equation. But this solution will obviously not contain an arbitrary constant, nor will it be included in the general integral (4).

It is easy to see that this is what was designated, Art. 86, as a *singular solution* of (1); for the equation

$$y=cx+\phi(c) \text{ .... }(4)$$

represents a family of ∞^1 straight lines. If they have an envelope, it will be found, as is well known, by eliminating c between the above equation and the equation obtained from the above by differentiating it with respect to c,

$$0=x+\phi'(c). \quad\text{..........................}(5)$$

But the result of eliminating c between (4) and (5) must be the same as the result of eliminating y' between the two equations

$$y=xy'+\phi(y'),$$
$$0=x+\phi'(y');$$

and hence the curve represented by the resulting equation will be the envelope of the family (4)—or a *singular solution* of the differential equation (1).

Of course the method of finding singular solutions by differentiation as here indicated might be employed in all cases, instead of the method of Art. 86; but by this method we usually find too much—not only the envelope of the curve-family, if one exists—but various other *loci* which may exist, composed of multiple points, cusps, etc.

The method of differentiation is generally the most simple for finding the singular solutions in the case of Clairaut's equation; since, of course, there can be no multiple points or cusps on one of the straight lines (3).

It is clear that a complete primitive representing a system of ∞^1 straight lines will always give rise to a differential equation of Clairaut's form—the singular solution of the equation being the curve to which the straight lines are tangent.

94. A more general equation of a form similar to (1) is

$$y=x\psi(y')+\phi(y'). \quad\text{.....................}(6)$$

Differentiate, regarding y' as a variable as well as x and y; hence

$$y'=\psi(y')+[x\psi'(y')+\phi'(y')]\frac{dy'}{dx},$$

or

$$\frac{dx}{dy'}+x\frac{\psi'(y')}{\psi(y')-y'}=\frac{\phi'(y')}{y'-\psi(y')}. \quad\text{..............}(7)$$

But this equation is linear in x, and may be treated by the method of Art. 68. If the general integral of (7) has been found in the form

$$\Omega(x, y', c)=0,$$

the elimination of y' between this equation and (6) will give the general integral of (6).

Example 1. Given

$$y=xy'+\frac{m}{y'}. \qquad\text{.................................(8)}$$

This is an example of Clairaut's equation. Differentiating, we find

$$0=\left(x-\frac{m}{y'^2}\right)\frac{dy'}{dx}.$$

From $$\frac{dy'}{dx}=0$$

or $$y'=c,$$

the general integral is seen to be

$$y=cx+\frac{m}{c}.$$

From $$x-\frac{m}{y'^2}=0$$

we have $$y'=\sqrt{\frac{m}{x}};$$

and this value of y', substituted in (8) gives as a singular solution

$$y^2=4mx.$$

Hence we see that the singular solution is the equation to a parabola, while the general integral represents its ∞^1 tangents.

Example 2. Given

$$x+yy'=ay'^2. \qquad\text{.................................(9)}$$

This is seen to be an equation of the form (6). Differentiating, we find, after eliminating y,

$$1+y'^2+\frac{ay'^2-x}{y'}\cdot\frac{dy'}{dx}=2ay'\frac{dy'}{dx};$$

or, corresponding to (7),

$$\frac{dx}{dy'}-\frac{x}{y'(1+y'^2)}=\frac{ay'}{1+y'^2}. \qquad\text{........................(10)}$$

The general integral of (10) is, by Art. 68,

$$x=\frac{y'}{\sqrt{1+y'^2}}\{c+a\log(y'+\sqrt{1+y'^2})\}\,;\ \ldots\ldots\ldots\ldots\ldots(11)$$

and the elimination of y' between (11) and (9) will give the general integral required.

EXAMPLES.

Integrate the following equations (1)–(5) by Sec. I. :—

(1) $x\frac{dy}{dx}-ay+y^2=x^{-2a}$.

(2) $x\frac{dy}{dx}-ay+y^2=x^{-\frac{2a}{3}}$.

(3) $x\frac{dy}{dx}-y+y^2=x^{\frac{2}{3}}$.

(4) $\frac{du}{dx}+u^2=cx^{-\frac{4}{3}}$.

(5) $\frac{du}{dx}+bu^2=cx^{-4}$.

Integrate the following equations (6)–(15) by Sec. II., giving also the singular solutions :—

(6) $y=xy'+\frac{m}{y'}$.

(7) $y=xy'+\sqrt{b^2+a^2y'^2}$.

(8) $y=xy'+y'-y'^3$.

(9) $y^2-2xyy'-1=y'^2(1-x^2)$.

(10) $y=(x-1)y'-y'^2$.

(11) $x^2y'^2-2(xy-2)y'+y^2=0$.

(12) $(y-xy')(ay'-b)=aby'$.

The singular solutions, as well as the general integrals of the differential equations to which the following geometrical problems give rise, are to be found and interpreted. For geometrical formulae, see Chap. IV., Examples.

(13) Determine a curve such that the sum of the intercepts made by the tangents on the axes of coordinates shall be constant and equal to a.

(14) Determine a curve such that the portion of its tangent intercepted between the axes of x and y shall be constant and equal to a.

(15) Determine a curve such that the area of the right triangle formed by its tangent with the axes shall be constant, and equal to a^2.

(16) Determine a curve such that the projection upon the axis of y of the perpendicular from the origin upon a tangent is constant, and equal to a.

CHAPTER VIII.

TOTAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND DEGREE IN THREE VARIABLES, WHICH ARE DERIVABLE FROM A SINGLE PRIMITIVE.

95. In this chapter we shall indicate how the integration of an ordinary differential equation in three variables of the form

$$P(x, y, z)dx+Q(x, y, z)dy+R(x, y, z)dz=0$$

may, under certain conditions which we shall develop, be made to depend upon the methods given in Chap. IV. for the ordinary differential equation of the first order and degree in two variables.

SECTION I.

The Genesis of the Total Differential Equation in Three Variables.

96. In Chap. I. it was seen that an equation of the form

$$\Omega(x, y, c)=0, \qquad (c=\text{const.})$$

when treated as a complete primitive always gives rise to an ordinary differential equation of the form

$$X(x, y)dy-Y(x, y)dx=0.$$

Similarly, if the equation in three variables,

$$F(x, y, z, c) = 0,$$

is given, we find by differentiation

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz = 0,$$

which, when c has been eliminated by means of $F = 0$, may be written

$$P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz = 0. \quad \text{...(1)}$$

Equation (1) is called a *total* differential equation—the word *total* being used in contradistinction to the word *partial* in the definition of a partial differential equation in three variables, Art. 15.

If the equation

$$F(x, y, z, c) = 0$$

be solved in terms of c in the form

$$\Omega(x, y, z) = c, \quad \text{.....................(2)}$$

we find, by differentiating (2),

$$d\Omega \equiv \frac{\partial \Omega}{\partial x}dx + \frac{\partial \Omega}{\partial y}dy + \frac{\partial \Omega}{\partial z}dz = 0. \quad \text{..............(3)}$$

The equation (3), which, for obvious reasons, is called an *exact* differential equation, must of course be equivalent to (1), since $\Omega = c$ is equivalent to $F = 0$. Hence, when an equation of the form (1) is given, if it is to be reducible to the exact form (3), certain conditions must hold. That is, there must obviously exist a function $\mu(x, y, z)$ such that,

$$\frac{\partial \Omega}{\partial x} = \mu P, \ \frac{\partial \Omega}{\partial y} = \mu Q, \ \frac{\partial \Omega}{\partial z} = \mu R; \quad \text{.............(4)}$$

and since

$$\frac{\partial^2 \Omega}{\partial y \partial x} = \frac{\partial^2 \Omega}{\partial x \partial y}, \text{ etc.,}$$

we obtain from (4),

$$\mu\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=Q\frac{\partial \mu}{\partial x}-P\frac{\partial \mu}{\partial y},$$

$$\mu\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)=R\frac{\partial \mu}{\partial y}-Q\frac{\partial \mu}{\partial z},$$

$$\mu\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)=P\frac{\partial \mu}{\partial z}-R\frac{\partial \mu}{\partial x}.$$

Multiplying the first of these equations by R, the second by P, and the third by Q, we find as the condition that (1) may have a general integral of the form (2),

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0. \quad \text{...(5)}$$

SECTION II.

Integration of the Total Differential Equation in Three Variables.

97. We saw in the last section that the condition (5) must be satisfied, in order that the total equation (1)

$$Pdx+Qdy+Rdz=0, \quad \text{..................(1)}$$

may be derivable from a primitive of the form

$$\Omega(x, y, z)=\text{const.}$$

It is clear that that condition will be satisfied if the variables in (1) can be separated in such manner that P, Q, and R contain only the variables x, y, and z, respectively; and the general integral of (1) will, in that case, obviously be given in the form

$$\int P(x)dx+\int Q(y)dy+\int R(z)dz=\text{const.}$$

Example. Given

$$yz\,dx+xz\,dy+xy\,dz=0.$$

The variables may here be separated by dividing by xyz; and we find

$$\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0.$$

The general integral is, therefore,

$$\log x+\log y+\log z=\text{const.},$$

or

$$xyz=c. \qquad (c=\text{const.})$$

98. The equation (1) may sometimes be rendered exact by an integrating factor which suggests itself. For instance, the equation

$$zy\,dx-zx\,dy-y^2dz=0,$$

for which the condition (5), Sec. 1, is satisfied, becomes exact when multiplied by the factor $\frac{1}{zy^2}$. We then have

$$\frac{y\,dx-x\,dy}{y^2}-\frac{dz}{z}=0,$$

of which the general integral is obviously

$$\frac{x}{y}-\log z=\text{const.}$$

99. If, however, the variables in (1) cannot be separated by inspection, and no integrating factor suggests itself, the usual method for integrating (1)—supposing the condition of integrability to be satisfied—is as follows:—

Since the condition (5), Sec. I., is satisfied, a general integral of (1) of the form

$$\Omega(x, y, z)=\text{const.} \quad\dots\dots\dots\dots(2)$$

exists, which, when totally differentiated, must give rise to an equation equivalent to (1). Suppose that one of the variables in (2), say z, is temporarily regarded as a constant; then the equation resulting from totally differentiating (2) will have the form

$$P\,dx+Q\,dy=0. \quad\dots\dots\dots\dots(6)$$

Now the general integral of the ordinary differential equation in two variables (6) will clearly *include* the general integral of (1), if in place of the arbitrary constant of integration an arbitrary function of z is introduced. In order to determine the *form* of this function of z, it is necessary to differentiate the general integral of (6), considering z also as variable, and compare the total differential equation thus resulting with (1). This will give rise to a second ordinary differential equation for determining the arbitrary function of z.

The choice as to *which* of the three variables is to be regarded as a constant will be determined by the form of the resulting equation (6). It may sometimes be easier to integrate, by the methods of Chap. IV., one of the two equations

$$Pdx+Rdz=0, \quad Qdy+Rdz=0,$$

which result from considering y and x respectively as constants, than to integrate (6).

Example. Integrate the equation

$$(ydx+xdy)(b-z)+xydz=0.$$

Here the condition (5), Sec. I., is found to be satisfied. If we assume z to be constant, as in this case is most convenient, the term $xydz$ vanishes, so that the given equation, after dividing by $(b-z)$, becomes

$$ydx+xdy=0.$$

The integral of this ordinary equation in two variables is

$$xy=\text{const.}\equiv\phi(z).$$

Differentiating, we find

$$ydx+xdy-\frac{d\phi}{dz}dz=0.$$

In order that this equation may be equivalent to the first one, we must have

$$-\frac{d\phi}{dz}=\frac{xy}{b-z},$$

or

$$\frac{d\phi}{dz}=\frac{\phi}{z-b}.$$

Here the variables are separate, so that an immediate integration gives

$$\phi(z) \equiv c(z-b). \qquad (c=\text{const.})$$

The required general integral is, therefore,

$$xy=c(z-b).$$

100. We shall give a method, based upon geometrical considerations, by means of which the total equation (1)

$$Pdx+Qdy+Rdz=0 \text{ ....(1)}$$

may, when the condition (5) of the last section is satisfied, be integrated by integrating *one* ordinary differential equation of the first order in two variables.

Since (5) is satisfied, (1) has a general integral of the form

$$\Omega(x, y, z)=c\,; \text{ ....(2)}$$

and (2) represents ∞^1 surfaces in space, called the *integral* surfaces of (1). If we cut these surfaces by a family of ∞^1 planes, say,

$$z=x+ay\,; \quad (a=\text{const.}) \text{ ....(3)}$$

then for each value of a we obtain ∞^1 curves of intersection of one of the planes with the ∞^1 surfaces (2); and these curves are represented by a differential equation in x and y. To find this differential equation, we only need to eliminate z and dz from (1) by means of (3) and of

$$dz=dx+ady\,;$$

giving the differential equation in the form

$$\phi(x, y, a)dx+\psi(x, y, a)dy=0. \text{ ....(4)}$$

If, now, (4) has been integrated, by the methods of Chap. IV.—a being an arbitrary constant—we may easily find the ∞^1 surfaces (2), since we know their ∞^2 curves of intersection with the planes (3). For the ∞^1 curves of intersection which pass through one point on the axis of the family of planes (3) will in general form one of the integral surfaces (2).

Now a point on the axis of the planes (2) is evidently determined by

$$y=0, \quad x=\kappa; \qquad (\kappa=\text{const.})$$

and if the general integral of (4) be

$$W(x, y, a)=\text{const.}, \qquad (5)$$

in order for the curves (5) to pass through the point $y=0$, $x=\kappa$ we must have

$$W(x, y, a)=W(\kappa, 0, a). \qquad (6)$$

When a varies, (6) represents the ∞^1 curves through the point $y=0$, $x=\kappa$; and if κ also varies we obtain successively the ∞^1 curves through each point on the axis of (3). That is, if by means of (3) we eliminate a from (6), we obtain the integral surfaces required in the form

$$W\left(x, y, \frac{z-x}{y}\right)-W\left(\kappa, 0, \frac{z-x}{y}\right)=0.$$

Thus the complete integration of (1) has been accomplished by integrating one ordinary differential equation, (4), in two variables. If the constant a happens to factor out of (4), some other family of planes must be used in place of (3).

This method is theoretically better than that of Art. 99, since only *one* differential equation in two variables has to be integrated; but the differential equation (4) is often more difficult to integrate than are equations (6) and (9) of Art. 99.

Example. Given

$$(y+z)dx+dy+dz=0.$$

This equation evidently satisfies (5), Sec. I.; and if we write

$$z=x+ay,$$

the equation (4) becomes

$$y'+y+\frac{1+x}{1+a}=0.$$

This is a linear equation, with the general integral

$$W(x, y, a) \equiv e^x\left(y + \frac{x}{1+a}\right) = c, \qquad (c = \text{const.})$$

Hence equation (6) has the form

$$e^x\left(y + \frac{x}{1+a}\right) - e^\kappa\left(0 + \frac{\kappa}{1+a}\right) = 0\,;$$

or, writing $\frac{z-x}{y}$ for a,

$$e^x \cdot \frac{y^2+yz}{y+z-x} - e^\kappa \frac{\kappa \cdot y}{y+z-x} = 0,$$

that is,

$$e^x(y+z) = \text{const.}$$

EXAMPLES.

Integrate the following ordinary differential equations in three variables, after verifying that the condition (5), Sec. I., is satisfied:

(1) $(y+z)dx + (z+x)dy + (x+y)dz = 0.$

(2) $xzdx + zy\,dy = (y^2+x^2)dz.$

(3) $(x-3y-z)dx + (2y-3x)dy + (z-x)dz = 0.$

(4) $ay^2z^2dx + bz^2x^2dy + cx^2y^2dz = 0.$

(5) $(y+a)^2dx + zdy - (y+a)dz = 0.$

(6) $(y^2+yz)dx + (xz+z^2)dy + (y^2-xy)dz = 0.$

(7) $(2x^2+2xy+2xz^2+1)dx + dy + 2z\,dz = 0.$

CHAPTER IX.

ORDINARY DIFFERENTIAL EQUATIONS OF THE SECOND ORDER IN TWO VARIABLES.

101. In this chapter we propose to develop a theory of integration for ordinary differential equations of the second order in two variables analogous to that developed in Chapter IV. for ordinary differential equations of the first order.

The *linear* differential equation of the second and higher orders will be treated separately in Chapter XI.

SECTION I.

Exact Differential Equations of the Second Order.

102. If an ordinary differential equation of the second order of the form

$$\Omega(x, y, y', y'')=0$$

be given, we know that the complete primitive, or general integral, is an equation involving two independent arbitrary constants, c_1, c_2, of the form

$$\Psi_1(x, y, c_1, c_2)=0.$$

It may be shown by means of the Theory of Functions that if the complete primitive of $\Omega=0$ be written in the form

$$y-W(x, c_1, c_2)=0;$$

and if

$$y-w(x, a_1, a_2)=0$$

be a *second* equation, which, when treated as a complete primitive, gives rise to the *same* differential equation of the second order, $\Omega=0$, then it must always be possible to choose the a_1, a_2 as such functions of c_1, c_2, say

$$a_1=\lambda_1(c_1, c_2),\ \ a_2=\lambda_2(c_1, c_2),$$

that

$$W(x, c_1, c_2)\equiv w(x, \lambda_1, \lambda_2).$$

If the above complete primitive $\Psi_1=0$ be differentiated, an equation of the form

$$\Psi_2(x, y, y', c_1, c_2)=0$$

will result, from which, by means of $\Psi_1=0$, c_1 and c_2 may be successively eliminated, giving rise to two independent differential equations of the first order of the form

$$\Phi_1(x, y, y', c_1)=0,\ \ \Phi_2(x, y, y', c_2)=0.$$

The elimination of y' between $\Phi_1=0$ and $\Phi_2=0$ would, of course, give $\Psi_1=0$ again.

Now let $\Phi_1=0$ be again differentiated, and c_1 be eliminated from the resulting differential equation of the second order by means of $\Phi_1=0$. By this means we must obtain the given differential equation of the second order, $\Omega=0$. This last equation must also, of course, be obtained by proceeding similarly with $\Phi_2=0$.

The differential equations of the first order, $\Phi_1=0$, $\Phi_2=0$, may, on the other hand, be considered as having been derived by an integration from $\Omega=0$; and for this reason, $\Phi_1=0$, $\Phi_2=0$ are said to be *first integrals* of the given differential equation of the second order, $\Omega=0$. From the manner of their genesis it is seen that there can be only two *independent* first integrals of $\Omega=0$, although the whole number of first integrals is infinite. That is, the constant of integration of any third first integral of $\Omega=0$, of the form

$$\Phi_3(x, y, y', c_3)=0$$

can always be expressed as a function of the constants of any two independent first integrals; otherwise, by eliminating y' from $\Phi_1=0$ and $\Phi_2=0$ by means of $\Phi_3=0$, two different forms of the complete primitive, $\Psi_1=0$ and $\Psi=0$, would be found, which is impossible.

For a method of expressing the general integral in the form of an infinite series, see Art. 122.

Example. It may be readily verified that the ordinary differential equation of the second order,

$$\frac{d^2y}{dx^2}+y=0,$$

has the first integrals

$$\left(\frac{dy}{dx}\right)^2+y^2=a^2,$$

$$\frac{dy}{dx}\cos x+y\sin x=b,$$

$$\frac{dy}{dx}\sin x-y\cos x=c,$$

$$\frac{dy}{dx}=y\cot(x+\alpha); \qquad (a^2,\ b,\ c,\ \alpha \text{ consts.})$$

but only *two* of them are independent. For, squaring and adding the second and third, and comparing with the first we find that a relation of the form

$$b^2+c^2=a^2$$

must exist. Also, developing $\cot(x+\alpha)$ in the fourth equation, and comparing the result with the sum of the second and third, we find

$$c+b\tan\alpha=0.$$

Hence, $$b=a\cos\alpha,\ \ c=-a\sin\alpha;$$

and only two of the four constants are independent.

The general integral of the original differential equation may be found either by integrating any one of the first integrals again by the methods of Chap. IV., or by eliminating $\frac{dy}{dx}$ between any two independent first integrals. Thus the result of eliminating $\frac{dy}{dx}$ between the second and the third of the above first integrals will be

$$y=b\sin x-c\cos x,$$

which is the general integral of the given differential equation of the second order.

Also, the first of the four first integrals may be written

$$\frac{dy}{\sqrt{a^2-y^2}}=dx,$$

which integrated gives

$$\sin^{-1}\frac{y}{a}=x+\beta,$$

or

$$\sin y=a\sin(x+\beta).$$

If now we put $m=a\cos\beta,\ n=a\sin\beta,$

the last equation reduces to

$$y=m\sin x+n\cos x,$$

which is evidently the general integral as found above. The same result may, of course, be found by eliminating $\frac{dy}{dx}$ between the first and fourth of the first integrals.

103. A differential equation of the second order

$$\Omega(x, y, y', y'')=0$$

is said to be *exact*, when the expression Ωdx is the exact differential of a function of the form $F(x, y, y')$ Since the equation $\Omega=0$ is derived from $F(x, y, y')=c$ by direct differentiation, y'' can only occur in the first degree. Otherwise $\Omega=0$ will *not* be an exact differential equation.

Thus the exact differential equation of the second order may be written

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}y'+\frac{\partial F}{\partial y'}y''=0, \qquad F\equiv F(x, y, y')\ldots(3)$$

In order to integrate this equation completely, we notice, first, that we can determine how the variable y' enters the function F by integrating the term $\frac{\partial F}{\partial y'}y''$ with respect to y'. If the expression thus found be differentiated totally with respect to x, and the result be subtracted from the first member of (3), the remainder must be a differential expression of an order not higher than the first. Also since this remainder is the difference of two exact differentials, it must itself be an exact differential, and y' can occur in it only to the first degree. Its integral, together with the terms already found by integrating with respect to y', will be the integral of the whole equation (3).

Example. The equation

$$xyy''+xy'^2-yy'=0, \quad \dots\dots\dots\dots\dots\dots\dots\dots(4)$$

is exact. For the term involving y'' is xyy'', and this being integrated with respect to y' gives xyy'. The last expression when differentiated totally with respect to x gives

$$xyy''+xy'^2+yy'.$$

Subtract this result from the first member of (4), and the remainder, $-2yy'$, will also be exact, having for its integral $-y^2$. Hence equation (4) is *exact*, and a first integral is

$$xyy'-y^2=c_1.$$

If now we divide (4) by x^2, it will be seen that a second first integral is

$$\frac{y}{x}y'=c_2.$$

Hence the complete primitive is

$$y^2=c_2x^2-c_1.$$

SECTION II.

Lie's Differential Equations of the Second Order.

104. Of course not every differential equation of the second order is exact; and there is no general method for integrating all such equations when not exact. It will be our object, however, to show in this paragraph, how the knowledge that the given differential equation of the second order *admits of*, or is *invariant under*, a given G_1 can be used to reduce the problem of integration in a number of the most important classes of differential equations of the second order. These invariant differential equations of the second order are sometimes called "*Lie's Equations of the Second Order.*"

105. Suppose that the infinitesimal transformation

$$Uf \equiv \xi(x, y)\frac{\partial f}{\partial x} + \eta(x, y)\frac{\partial f}{\partial y}$$

of a given G_1 in two variables be twice extended by Art. 47; in the four variables x, y, y', y'' the twice-extended infinitesimal transformation will have the form

$$U''f \equiv \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'} + \eta''\frac{\partial f}{\partial y''},$$

where $$\eta' \equiv \frac{dy}{dx} - y'\frac{d\xi}{dx}, \quad \eta'' \equiv \frac{dy'}{dx} - y''\frac{d\xi}{dx}.$$

The necessary and sufficient condition that an equation in the four variables x, y, y', y'', of the form

$$\Omega(x, y, y', y'') = 0,$$

may be *invariant* under the G_1 $U''f$ is, by Art. 42, that the expression $U''(\Omega)$ shall be zero, either identically, or by means of $\Omega = 0$. It was also shown in Art. 43 that if u, v, and w be three independent solutions of the linear partial differential equation

$$U''f \equiv \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'} + \eta''\frac{\partial f}{\partial y''} = 0,$$

the most general form of the invariant equation, $\Omega = 0$, is obtained when Ω is expressed as an arbitrary function of u, v, and w, say

$$\Omega(x, y, y', y'') \equiv \mathrm{F}(u, v, w) = 0.$$

Or, if we choose, we may solve $\mathrm{F} = 0$ in terms of one of the three quantities u, v, or w, say in terms of w, and thus put the most general invariant equation in the form

$$w - \Phi(u, v) = 0.$$

But x and y may be interpreted as point coordinates in a plane; and y' and y'' as the differentials

$$\frac{dy}{dx}, \ \frac{d^2y}{dx^2},$$

respectively. In this case the equation $\Omega=0$ is a differential equation of the second order in the variables x and y; and if the expression $U''(\Omega)$ is zero, either identically or by means of $\Omega=0$, the differential equation $\Omega=0$ is said to be *invariant* under, or to *admit of*, the G_1 $U''f$.

Now the differential equation $\Omega=0$ represents a doubly infinite system of curves in the plane; and that the equation $\Omega=0$ shall be invariant under $U''f$ means that the system of curves must be invariant under $U''f$ also. For, if we designated the new variables, as usual, by x_1, y_1, y_1', y_1'', and the transformed equation by $\Omega_1=0$, then, by hypothesis, Ω_1 must have the same form in the new variables that Ω had in the variables x, y, y', y'': that is, $\Omega_1=0$ must represent the same family of curves that $\Omega=0$ represented.

But this family of curves will also be transformed by

$$Uf\equiv\xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y},$$

and at any point of general position (x, y), y' and y'' will receive the *same* increments, Art. 47, by means of Uf, that they receive at that point by means of $U''f$. Hence, a point P, in the plane which satisfies the system of values x, y, y', y'', will always be transformed by means of Uf to the point P_1 which satisfies the system of transformed values x_1, y_1, y_1' y_1''; and it is clear that since P passes to the same position P_1 under the transformation Uf that it does under $U''f$, and since y' and y'' are transformed by Uf exactly as they are by $U''f$, the family of curves, represented by $\Omega=0$, must also be invariant under Uf. Thus it is clear that the condition that a differential equation of the second order

$$\Omega(x, y, y', y'')=0$$

shall be invariant under the twice-extended G_1 $U''f$, is the same as that the family of ∞^2 integral-curves of $\Omega=0$ shall be invariant under Uf.

When the condition that $U''(\Omega)$ is zero, either identically or by means of $\Omega=0$, is satisfied, we sometimes say, for brevity, that $\Omega=0$ is invariant under the G_1 Uf, instead of under the twice-extended G_1 $U''f$.

106. We have a general method, Art. 23, for determining the quantities u, v and w of the preceding article, and we have seen, Art. 56, that when u has been found from the differential equation

$$\frac{dx}{\xi}=\frac{dy}{\eta}, \qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

then v can be found as the second integral-function of the system

$$\frac{dx}{\xi}=\frac{dy}{\eta}=\frac{dy'}{\eta'}, \qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

by means of a quadrature. Unless the path-curves of the G_1, which are represented by

$$u=\text{const.},$$

are known, it will be necessary to perform an integration to find u from the above differential equation of the first order (1); but we shall show that when u and v have been found, w can be found by mere processes of differentiation. Also w, as will be seen, must necessarily contain y''; and as $v(x, y, y')$ was called, Art. 37, a differential invariant, of the given G_1, of the first order, so $w(x, y, y', y'')$, for reasons which are obvious, is called a *differential invariant of the given* G_1, *of the second order.*

107. We shall now show that when u and v have the meaning of the last article assigned to them, and are known, that w, the third independent integral-function of the system

$$\frac{dx}{\xi}=\frac{dy}{\eta}=\frac{dy'}{\eta'}=\frac{dy''}{\eta''}, \qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

can be found by differentiation.

To this end let a and b be any two constants. Then

$$v-au-b=0 \quad\text{.........................}(4)$$

is a differential equation of the first order which is invariant under $U''f$, since

$$U''(v-au-b)\equiv 0.$$

If now a is supposed to retain a fixed value, while b varies, (4) represents ∞^1 differential equations of the first order which are invariant under $U''f$. Each of these differential equations represents ∞^1 integral curves in the plane, so that there are ∞^1 families, each of ∞^1 curves, which as a system are invariant. This system of ∞^2 curves must be represented by a differential equation of the second order, which must also be invariant under $U''f$. We obtain this differential equation of the second order by differentiating

$$v(x, y, y')-au(x, y)=b$$

totally with respect to x. Hence, we find,

$$\frac{dv}{dx}-a\frac{du}{dx}=0, \quad\text{........................}(5)$$

or

$$\frac{\dfrac{\partial v}{\partial x}+\dfrac{\partial v}{\partial y}y'+\dfrac{\partial v}{\partial y'}y''}{\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}y'}-a=0; \quad\text{...............}(5')$$

or, briefly,

$$w(x, y, y', y'')-a=0. \quad\text{................}(6)$$

Since (6) is invariant under the G_1, we must have

$$U''(w-a)=0,$$

by means of $w-a=0$. But, since a is a constant,

$$U''(w-a)\equiv U''(w),$$

that is,

$$U''(w)\equiv 0;$$

and w is a solution of the linear partial differential equation

$$U''f=0.$$

From (5′) we see that since v must contain y', w must also contain y''; hence we can take this function to be the third solution of $U''f=0$, that is, the third integral of the system (3). It is seen further from (5) that

$$w \equiv \frac{dv}{du};$$

so that the most general invariant differential equation of the second order may always be written in the form

$$\frac{dv}{du} - \Phi(u, v) = 0. \quad \text{......................}(7)$$

108. The complete integration of (7) may be accomplished by the integration of an ordinary differential equation of the first order in two variables, together with one quadrature. For (7), as is seen by its form, may be considered a differential equation of the first order in the variables u and v, and if (7) has been integrated, say in the form

$$v = \phi(u, a), \quad \text{.........................}(8)$$

this will be a differential equation of the first order in x and y, since v contains y'. But since u and v are solutions of

$$U'f \equiv \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'} = 0,$$

it is clear that the equation (8) must admit of the $G_1 U'f$. Hence, as we know, Art. 59, (8) may be integrated by a quadrature.

Summing up the above results it is seen that the complete integration of a given differential equation of the second order, which admits of a known G_1, may be performed as follows:

A differential equation of the first order must be integrated to find u—unless the path-curves $u=const.$ of the G_1 happen to be known—then a quadrature is necessary to find v; then the integration of a differential *equation of the first order between u and v; and finally a quadrature.*

A method which is theoretically an improvement upon this one is given Art. 163.

SECTION III.

Classes of Lie's Differential Equations of the Second Order.

109. We shall now apply the results of the last paragraph to find the classes of differential equations of the second order, which are invariant under some of the most common G_1's in the plane, and which are therefore integrable by the above methods.

110. *To find all differential equations of the second order which are invariant under the G_1 of translations along the x-axis.*

Here the G_1 has the form

$$Uf \equiv \frac{\partial f}{\partial x};$$

and it is seen at once that $\eta' \equiv \eta'' \equiv 0$, that is, the twice extended G_1 has the form

$$U''f \equiv \frac{\partial f}{\partial x}.$$

Thus it is necessary to find three independent integral-functions, u, v, w, of the simultaneous system

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dy'}{0} = \frac{dy''}{0}.$$

It is obvious that we may assume

$$u \equiv y, \quad v \equiv y', \quad w \equiv \frac{dv}{du} \equiv \frac{y''}{y'};$$

or

$$y \equiv u, \; y' \equiv v, \quad y'' \equiv v\frac{dv}{du}.$$

The most general invariant differential equation of the second order has therefore the form

$$\mathrm{F}\left(y, y', \frac{y''}{y'}\right)=0,$$

where, of course, since y' is itself an integral-function of the above simultaneous system, y'' might be written in place of $\frac{y''}{y'}$; that is, any differential equation of the second order which does not contain the variable x is invariant under the translations along the x-axis.

If $\mathrm{F}=0$ be written in the form

$$\frac{y''}{y'}-\Phi(y, y')=0,$$

or, Art. 107,

$$\frac{dv}{du}-\Phi(u, v)=0,$$

it is seen that we obtain a differential equation of the first order in u and v. Its integration will give an equation of the form

$$v=\psi(u, c_1), \qquad (c_1=\text{const.})$$

or

$$y'=\Psi(y, c_1).$$

This equation must also admit of the given G_1, Art. 108; hence by Art. 59 a quadrature gives the general integral sought in the form

$$\int\frac{dy}{\psi(y, c_1)}-x=c_2. \qquad (c_2=\text{const.})$$

If $\mathrm{F}=0$ has the special form

$$y''-\Phi(y)=0,$$

it is only necessary to assume

$$y\equiv u, \;\; y''\equiv v\frac{dv}{du},$$

and the equation may be integrated by two quadratures.

Example. Given $yy'' - y'^2 = 0.$
Here, according to the above method, we put

$$y'' = v\frac{dv}{du},\ y' = v,\ y = u,$$

in order to obtain the differential equation of the first order between u and v.

Hence

$$uv\frac{dv}{du} - v^2 = 0,$$

or

$$\frac{dv}{du} = \frac{v}{u}.$$

A quadrature now gives $\frac{v}{u} = c_1,$

or

$$\frac{y'}{y} = c_1;$$

whence, by a second quadrature,

$$y = c_2 . e^{c_1 x}.$$

111. In a manner entirely analogous to that of the last article we may find the differential equations of the second order which are invariant under the G_1 of translations along the y-axis. Since

$$Uf \equiv \frac{\partial f}{\partial y},$$

it is seen at once that we may write

$$u \equiv x,\ v \equiv y',\ w \equiv \frac{dv}{du} \equiv y''.$$

Thus all differential equations of the second order,

$$\mathrm{F}(x, y', y'') = 0$$

which do not contain the variable y, are invariant under the translations along the y-axis.

If $\mathrm{F} = 0$ be written in the form

$$y'' - \Phi(x, y') = 0,$$

or

$$\frac{dv}{du} - \Phi(u, v) = 0,$$

it is seen that this is a differential equation of the first order in u and v. Its integration will give an equation of the form

$$v=\psi(u, c_1),$$

or

$$y'=\psi(x, c_1),$$

so that a quadrature will give the general integral required in the form

$$y=\int\psi(x, c_1)\,dx+c_2.$$

Example. Given $(1-x^2)y''-xy'-2=0.$

This equation, not containing the variable y, admits of

$$Uf\equiv\frac{\partial f}{\partial y}.$$

Hence we must substitute

$$x\equiv u,\ y'\equiv v,\ y''\equiv\frac{dv}{du};$$

so that there results $(1-u^2)\dfrac{dv}{du}-uv-2=0,$

or

$$\frac{dv}{du}-\frac{u}{1-u^2}\cdot v-\frac{2}{1-u^2}=0.$$

This differential equation of the first order in u and v is *linear*; its general integral is given by Art. 68 in the form

$$v=\frac{2}{\sqrt{1-u^2}}(\sin^{-1}u+c_1),$$

or

$$y'=\frac{2}{\sqrt{1-x^2}}(\sin^{-1}x+c_1).$$

This equation in x and y' must also admit of Uf, Art. 108, so that another quadrature gives the required general integral in the form

$$y=(\sin^{-1}x)^2+2c_1\sin^{-1}x+c_2.$$

112. It may be remarked that the equation

$$y''=\psi(y'), \quad\text{.........................(1)}$$

that is, the general differential equation of the second order in which *neither* of the variables is present, admits of both of the G_1's

$$U_1f\equiv\frac{\partial f}{\partial x},\ \ U_2f\equiv\frac{\partial f}{\partial y}.$$

We find from (1) by a quadrature

$$\int\frac{dy'}{\psi(y')}=x+c_1, \qquad (c_1=\text{const.})$$

or, say $$y'=w(x+c_1);$$

whence, by a second quadrature,

$$y=\int w(x+c_1)dx+c_2.$$

113. *To find all differential equations of the second order which are invariant under the G_1 of affine transformations*

$$Uf\equiv x\frac{\partial f}{\partial x}.$$

The twice-extended G_1 is

$$U''f=x\frac{\partial f}{\partial x}-y'\frac{\partial f}{\partial y'}-2y''\frac{\partial f}{\partial y''};$$

while the simultaneous system to be integrated is

$$\frac{dx}{x}=\frac{dy}{0}=\frac{dy'}{-y'}=\frac{dy''}{-2y''}.$$

It is evident that we may assume, Art. 63,

$$u=y,\ \ v=xy';$$

so that $$w\equiv\frac{dv}{du}\equiv\frac{xdy'+y'dx}{dy}=\frac{xy''+y'}{y'}.$$

The required differential equation of the second order has therefore the form

$$\frac{xy''+y'}{y'}-\Phi(y,xy')=0,$$

or $$\frac{d(xy')}{dy}-\Phi(y,xy')=0.$$

By integrating this differential equation of the first order, a result of the form

$$xy'-w(y,c_1)=0 \qquad (c_1=\text{const.})$$

is found; and since this last differential equation of the first order is known, Art. 108, to be invariant under

$$Uf \equiv x\frac{\partial f}{\partial x},$$

a quadrature will give the general integral required in the form

$$y = \Omega(x, c_1, c_2) = 0. \qquad (c_2 = \text{const.})$$

114. In an analogous manner the most general differential equation of the second order which is invariant under the G_1

$$Uf \equiv y\frac{\partial f}{\partial y},$$

since $u \equiv x$, $v \equiv \frac{y'}{y}$, will be found to have the form

$$\frac{d\frac{y'}{y}}{dx} - \Phi\left(x, \frac{y'}{y}\right) = 0,$$

or

$$\frac{y''y - y'^2}{y^2} - \Phi\left(x, \frac{y'}{y}\right) = 0.$$

It should be noticed that the so-called *abridged linear equation* of the second order of the form

$$y'' + X_1(x)y' + X(x)y = 0,$$

where X_1 and X are functions of x alone, is a particular case of the general differential equation of the second order which is invariant under

$$Uf \equiv y\frac{\partial f}{\partial y}.$$

For the above invariant equation may obviously be written

$$\frac{y''}{y} - \Psi\left(x, \frac{y'}{y}\right) = 0,$$

when we assume

$$\left(\frac{y'}{y}\right)^2+\Phi\left(x,\frac{y'}{y}\right)\equiv\Psi\left(x,\frac{y'}{y}\right).$$

If now we suppose Ψ to have the special form

$$\Psi\left(x,\frac{y'}{y}\right)\equiv -X_1(x).\frac{y'}{y}-X(x),$$

the invariant equation will assume the form of the abridged linear equation.

A further discussion of this equation will be given in Chapter XI.

115. *To find all differential equations of the second order which are invariant under the* G_1

$$Uf\equiv\phi(x)\frac{\partial f}{\partial y}.$$

Here it is readily seen that

$$u\equiv x,\ v\equiv\phi y'-\phi' y,\ \frac{dv}{du}\equiv\phi y''-\phi'' y,$$

where ϕ' and ϕ'' are written for

$$\frac{d\phi}{dx},\ \frac{d^2\phi}{dx^2}$$

respectively. Thus the most general invariant differential equation is

$$\phi y''-\phi'' y-\Phi(x,\ \phi y'-\phi' y)=0.$$

If ϕ is an integral-function of the abridged linear equation of the second order

$$y''+X_2(x)y'+X_1(x)y=0,$$

that is, if ϕ satisfies the identity

$$\phi''+X_2(x)\phi'+X_1(x)\phi\equiv 0\,;$$

then the *general* linear equation of the second order

$$y''+X_2(x)y'+X_1(x)y+X_0(x)=0$$

will be a particular case of the above invariant differential equation.

For we only need assume Φ in the form

$$\Phi \equiv -X_2(x)(\phi y' - \phi' y) - X_0(x)\phi,$$

when the invariant equation becomes

$$y'' + X_2(x)y' - \left(\frac{\phi'' + X_2\phi'}{\phi}\right)y + X_0 = 0;$$

that is, since ϕ satisfies the abridged equation

$$y'' + X_2(x)\ y' + X_1(x)y + X_0(x) = 0.$$

116. *To find all differential equations of the second order which are invariant under the G_1 of perspective transformations*

$$Uf \equiv x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}.$$

It is seen from Art. 64 that we may here assume

$$u = \frac{y}{x},\ u = y',$$

so that

$$\frac{dv}{du} \equiv \frac{xy''}{y' - \dfrac{y}{x}}.$$

Thus the most general invariant differential equation of the second order is

$$\frac{xy''}{y' - \dfrac{y}{x}} - \Phi\left(\frac{y}{x}, y'\right) = 0;$$

or, as it may obviously be written,

$$\mathrm{F}\left(y', xy'', \frac{y}{x}\right) = 0.$$

The integration of the differential equation of the first order in $u \equiv \frac{y}{x}$ and $v \equiv y'$,

$$\frac{dy'}{d\frac{y}{x}} - \Phi\left(\frac{y}{x}, y'\right) = 0,$$

will give a result of the form

$$y' - \phi\left(\frac{y}{x}, c\right) = 0.$$

This equation must, of course, admit of Uf, and hence its general integral, and thus the general integral of $F=0$, may now be found by a quadrature.

Example. Given the differential equation

$$xyy'' - xy'^2 + yy' = 0.$$

This is obviously an equation of the form $F=0$. Hence it may be written in the form

$$\frac{xy''}{y' - \frac{y}{x}} - \Phi\left(\frac{y}{x}, y'\right) = 0\,;$$

and, in fact, we have

$$\frac{xy''}{y' - \frac{y}{x}} - \frac{xy'}{y} = 0.$$

Hence

$$\frac{dv}{du} = \frac{v}{u}\,;$$

that is,

$$v = c_1 u,$$

or

$$y' = \frac{c_1 y}{x}.$$

A quadrature will now give the general integral required in the form

$$y = c_2 \,.\, x^{c_1}.$$

117. The values of u and v for the groups

$$-y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y},\ x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y},\ x\frac{\partial f}{\partial y},\ y\frac{\partial f}{\partial x},\ x^2\frac{\partial f}{\partial x} + xy\frac{\partial f}{\partial y},$$

are given Arts. 66-73. It will be a valuable exercise for the reader to find the corresponding invariant differential equations, or Lie's equations, of the second order.

118. The criterion that a given differential equation of the second order,

$$\Omega(x, y, y', y'')=0,$$

shall admit of a given twice-extended G_1,

$$Uf \equiv \xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y}+\eta'\frac{\partial f}{\partial y'}+\eta''\frac{\partial f}{\partial y''},$$

is that the expression

$$U''(\Omega) \equiv \xi\frac{\partial \Omega}{\partial x}+\eta\frac{\partial \Omega}{\partial y}+\eta'\frac{\partial \Omega}{\partial y'}+\eta''\frac{\partial \Omega}{\partial y''}$$

shall be zero, either identically or by means of $\Omega=0$. It is often possible to find, from this condition, the G_1 of which the given equation admits, as is best illustrated by an example.

Example. The condition that the equation

$$\Omega \equiv yy''-xy'y''-4y'^2=0$$

shall admit of Uf, is that the expression

$$U''(\Omega) \equiv -\xi y'y''+\eta y''-\eta' xy''-8y'\eta'+y\eta''-xy'\eta''$$

shall be zero identically, or by means of $\Omega=0$. On comparing this expression with $\Omega=0$, we see that for $\xi \equiv 0$, $\eta \equiv y$, and thus $\eta' \equiv y'$, $\eta'' \equiv y''$, we have

$$U''(\Omega) \equiv 2\Omega.$$

The given equation must therefore admit of $y\frac{\partial f}{\partial y}$, and hence be one of the type

$$\frac{yy''-y'^2}{y^2}-\Phi\left(x, \frac{y'}{y}\right)=0;$$

and, in fact, it may be written

$$\frac{yy''-y'^2}{y^2}-\frac{3yy'^2+xy'^3}{y^2(y-xy')}=0.$$

Now, Art. 114, assume $u=x$, $v=\frac{y'}{y}$,

and the last equation takes the form

$$\frac{dv}{du}-\frac{v^2(3+uv)}{1-uv},$$

or

$$(1-uv)udv-(3uv+u^2v^2)vdu=0,$$

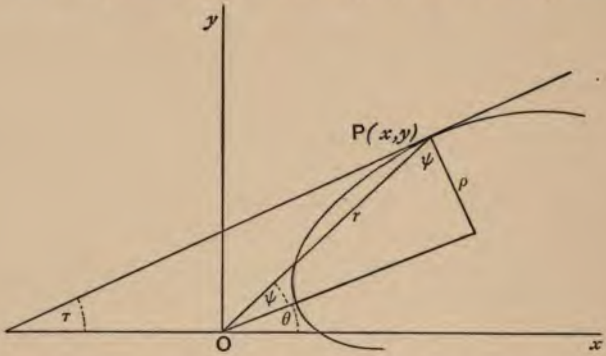
an equation of the first order which may be integrated, Art. 67, by a quadrature. Since the resulting equation will admit of the G_1,

$$Uf \equiv y\frac{\partial f}{\partial y},$$

a second quadrature will give us the general integral of the given differential equation of the second order, $\Omega = 0$.

119. We shall apply the theory of integration of this paragraph to an example involving geometrical considerations.

A family of curves in the plane is often defined by an equation expressing a relation between such magnitudes



as the subtangents, the radii vectores, the perpendiculars from the origin on the tangents, etc., giving rise by that means to a differential equation of a certain order.

For example, suppose it is required to find all curves which are defined by a relation between the line r connecting the point (x, y) with the origin; the angle ψ between this line and the radius of curvature, ρ; and the radius of curvature itself. Such a relation is given by an equation of the form

$$\Phi(r, \psi, \rho) = 0.$$

It is geometrically evident that such a curve will pass, by means of a rotation, into a congruent one. In other words, the family of curves represented by $\Phi=0$ admit of the G_1 of rotations

$$Uf\equiv -y\frac{\partial f}{\partial x}+x\frac{\partial f}{\partial y},$$

where, of course, the values of r, ψ, and ρ, in $\Phi=0$, are to be expressed in Cartesian coordinates.

Since, in the variables x and y, $\Phi=0$ will be a differential equation of the second order, and since we know the path-curves of Uf, it is clear that, by Art. 108, the above family of ∞^2 curves may be determined by the integration of a differential equation of the first order, together with a quadrature.

120. In integrating the following differential equations of the second order, the equations should first be examined to see whether they are *exact* or not, Art. 103. If an equation is exact, its integration may, Art. 103, be reduced to the integration of a differential equation of the first order.

If the differential equation of the second order is *not* exact, it should be compared with the types of invariant equations, Arts. 110–117; and if the equation belongs to one of those types, its integration is to be accomplished by the method indicated for that type.

If the equation is not exact, and does not belong to one of the given types, the G_1 of which it admits is to be sought by the method of Art. 118.

In examples (1)–(25) it should be verified geometrically, whenever practicable, that the family of ∞^2 integral curves found admit of the G_1 which was used in integrating the differential equation. Thus, it is geometrically evident that the integral curves

$$x^2+2c_1x+y^2+2c_2y=0$$

of Ex. 26, which are the ∞^2 circles through the origin, admit of the G_1 of rotations.

We give, for convenience of reference, a table of some of the more important invariant differential equations of the second order.

Group of One Parameter.	*Type of Invariant Differential Equation.*
(1) $Uf \equiv \frac{\partial f}{\partial x}$,	(1) $F(y, y', y'') = 0.$

This invariant equation *includes*, of course, the types

$$F(y, y'') = 0, \quad F(y', y'') = 0.$$

(2) $Uf \equiv \frac{\partial f}{\partial y}$, (2) $F(x, y', y'') = 0.$

This equation includes

$$F(x, y'') = 0, \quad F(y', y'') = 0.$$

(3) $Uf \equiv x\frac{\partial f}{\partial x}$, (3) $F\left(y, xy', \frac{xy''}{y'}\right) = 0.$

(4) $Uf \equiv y\frac{\partial f}{\partial y}$, (4) $F\left(x, \frac{y'}{y}, \frac{y''}{y}\right) = 0.$

(5) $Uf \equiv x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$, (5) $F\left(\frac{y}{x}, y', xy''\right) = 0.$

EXAMPLES.

(1) $yy'' + y'^2 = 1.$
(2) $(1 - x^2)y'' - xy' + y = 0.$
(3) $y'' = xe^x.$
(4) $y'' = x + \sin x.$
(5) $y'' = \frac{-a}{(2ax - x^2)^{\frac{3}{2}}}.$
(6) $y'' = \pm a^2 y.$
(7) $y^3 y'' = a.$
(8) $y'' = \frac{1}{\sqrt{ay}}.$
(9) $a^2 y''^2 = 1 + y'^2.$
(10) $y'' = y'^2 + 1.$
(11) $y'' = -(y'^2 + 1).$
(12) $a^2 y''^2 = (1 + y'^2)^3.$
(13) $xy'' + y' = 0.$
(14) $y' - xy'' = f(y'').$
(15) $(1 + x^2)y'' + 1 + y'^2 = 0.$
(16) $(1 - x^2)y'' + xy' = x.$
(17) $xy'' + y' = x^n.$
(18) $(a^2 - x^2)y'' - \frac{a^2}{x}y' + \frac{x^2}{a} = 0.$
(19) $y'' + yy' = 0.$
(20) $y(1 - \log y)y'' + (1 + \log y)y'^2 = 0.$
(21) $y'' = \frac{2y}{x^2}.$
(22) $xy'' - xy'^2 + y' = 0.$
(23) $x^2 y'' - xy' - 3y = 0.$
(24) $xy'^2 + xyy'' - yy' = 0.$
(25) $x^2 y'' - xy' + y = 0.$
(26) $(x^2 + y^2)y'' - 2xy'^3 + 2yy'^2 - 2xy' + 2y = 0.$

The following geometrical examples lead to ordinary differential equations of the second order which admit of a G_1. The formulae for a plane curve given, Art. 73, together with the two given below, are convenient for reference.

Differential of an arc s, measured from a fixed point on the curve up to a point (x, y),

$$\frac{ds}{dx}=\sqrt{1+y'^2}.$$

Radius of Curvature, $$\mp\frac{(1+y'^2)^{\frac{3}{2}}}{y''}.$$

(27) Determine a curve such that the length of the arc measured from a fixed point on it is equal to the intercept of the tangent on the axis of x.

(28) Show that the curve whose radius of curvature is proportional to the cube of the normal is a conic section.

(29) Required the family of curves in which the radius of curvature is constant and equal to a.

(30) Determine the family of curves in which the radius of curvature is equal to the normal (a) when the two have the same direction, (b) when the two have opposite directions.

(31) Determine the family of curves in which the radius of curvature is equal to twice the normal (a) when the two have the same direction, (b) when the two have opposite directions.

(32) Show that if τ is the angle which the tangent at any point (x, y) on a given curve makes with the x-axis (see figure, Art. 119), while ρ is the radius of curvature at that point, all curves defined by a relation of the form

$$\Omega(x, \tau, \rho)=0$$

may be determined by the integration of a differential equation of the first order and a quadrature.

(33) If the angle between the x-axis and the line joining the centre of curvature with the origin be designated by θ, and if ϕ have the meaning of Art. 119, show that all curves defined by a relation of the form

$$\Omega(\phi, \tau, \theta)=0$$

between the three angles ϕ, τ, θ may be determined by the integration of a differential equation of the first order and a quadrature.

CHAPTER X.

THE DIFFERENTIAL EQUATION OF THE m^{th} ORDER IN TWO VARIABLES.

121. In this chapter we propose to indicate briefly how the methods of Chapters IV. and IX. for integrating invariant differential equations of the first and second orders may be extended to equations of an order higher than the second.

SECTION I.

Lie's Differential Equations of the m^{th} Order.

122. A differential equation of the m^{th} order in two variables has the general form

$$\Omega(x, y, y', \dots, y^{(m)}) = 0,$$

where Ω must, of course, actually contain $y^{(m)}$; and the complete primitive, or general integral, (Art. 5), of this equation has the general form

$$\Psi(x, y, c_1, \dots, c_m) = 0,$$

where $c_1, \dots, c_m$ are m independent arbitrary constants.

We saw in Art. 72 that the general integral of an ordinary differential equation of the first order in two variables may always be expressed in the form of an infinite series involving one arbitrary constant. We shall now show that the general integral of the ordinary differential equation of the m^{th} order in two variables may, similarly, be expressed as an infinite series involving m

arbitrary constants—although, as in Art. 72, we shall not investigate the question of whether this series always converges or not. It may be remarked that this method for obtaining the general integral is of little *practical* value for such equations as are neither linear (Chap. XI.) nor reducible to a linear form. If

$$y^{(m)}=\mathrm{F}_1(x, y, y', \dots y^{(m-1)}) \;\dots\dots\dots\dots\dots\dots\dots\dots (1)$$

be the given differential equation, by differentiation we find $y^{(m+1)}$ as a function of x and y, and the differential coefficients up to the m^{th}, if F_1 actually contains $y^{(m-1)}$. Substituting, in the result, for $y^{(m}$ its value as given by (1), we find for $y^{(m+1)}$,

$$y^{(m+1)}=\mathrm{F}_2(x, y, y', \dots y^{(m-1)}). \;\dots\dots\dots\dots\dots\dots\dots (2)$$

Differentiating (2), and reducing as before, we find

$$y^{(m+2)}=\mathrm{F}_3(x, y, y', \dots y^{(m-1)}) ; \;\dots\dots\dots\dots\dots\dots\dots (3)$$

and proceeding in this way we see that all differential coefficients of y of an order higher than the m^{th} are expressible in terms of $x, y, y', \dots, y^{(m-1)}$, by means of (1).

Now when we assign to x the initial value x_0, let the corresponding initial values of $y, y', \dots, y^{(m-1)}$ be represented by $y_0, y_0', \dots, y_0^{(m-1)}$, while the second members of (1), (2), (3), … become $\mathrm{F}_1^{(0)}, \mathrm{F}_2^{(0)}, \mathrm{F}_3^{(0)}, \dots$ respectively ; then, by Taylor's theorem, we find, as in Art. 72,

$$y=y_0+y_0'(x-x_0)+\frac{y_0''}{1\,.\,2}(x-x_0)^2+\dots+\frac{y_0^{(m-1)}}{1\,.\,2\dots(m-1)}(x-x_0)^{m-1}+\dots$$
$$+\frac{\mathrm{F}_1^{(0)}}{1\,.\,2\dots m}(x-x_0)^m+\frac{\mathrm{F}_2^{(0)}}{1\,.\,2\dots(m+1)}(x-x_0)^{(m+1)}+\dots. \quad (4)$$

In this expression for the general integral of (1), x_0 is a special numerical value of x ; and, as the $\mathrm{F}_i^{(0)}$ are functions of the m arbitrary constants $y_0, y'_0, \dots y_0^{(m-1)}$, (4) really contains only m independent arbitrary constants.

123. We shall now give a method for finding all differential equations of the m^{th} order, $\Omega=0$, which are invariant under a given G_1. Such equations are sometimes designated as "*Lie's equations of the m^{th} order.*"

In order to find all differential equations of the second order which are invariant under a given G_1

$$Uf\equiv\xi(x, y)\frac{\partial f}{\partial x}+\eta(x, y)\frac{\partial f}{\partial y},$$

we saw that, if

$$U''f \equiv \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y} + \eta''\frac{\partial f}{\partial y''}$$

represent the twice-extended G_1, it is necessary to find three independent solutions of the linear partial equation in four variables

$$U''f = 0.$$

If these solutions are represented by

$$u(x, y),\ v(x, y, y'),\ \frac{dv}{du} \equiv w(x, y, y', y''),$$

then the most general invariant differential equation of the second order is

$$\mathrm{F}(u, v, w) = 0.$$

In an entirely analogous manner we may see that to find the most general differential equation of the third order which is invariant under Uf, it is necessary to find four independent solutions of the linear partial differential equation in five variables

$$U'''f = 0,$$

where $U'''f$ is the thrice-extended G_1, so that

$$U'''f \equiv \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'} + \eta''\frac{\partial f}{\partial y''} + \eta'''\frac{\partial f}{\partial y'''}.$$

Just as we saw in Art. 107 that we could assume

$$w \equiv \frac{dv}{du},$$

so now, if u, v, w be *three* solutions of $U'''f = 0$, the function

$$\frac{dw}{du} \equiv \frac{d^2v}{du^2}$$

may be seen to be a *fourth* solution of $U'''f = 0$, which must contain y'''. Hence the most general invariant differential equation of the third order has the form

$$\mathrm{F}\left(u, v, \frac{dv}{du}, \frac{d^2v}{du^2}\right)=0.$$

Unless the path-curves of the G_1

$$u=\text{const.}$$

are known, it will be necessary to integrate a differential equation of the first order in two variables

$$\frac{dx}{\xi}=\frac{dy}{\eta}$$

to find u. Then v, Art. 56, may be found by a quadrature, and of course the other two solutions

$$\frac{dv}{du}, \frac{d^2v}{du^2}$$

by mere differentiations.

It is obvious now that the most general invariant differential equation of the m^{th} order will have the form

$$\mathrm{F}\left(u, v, \frac{dv}{du}, \frac{d^2v}{du^2}, \cdots \frac{d^{m-1}v}{du^{m-1}}\right)=0;$$

and it is clear that, in the most unfavourable case, this differential equation may be established by the integration of a differential equation of the first order in two variables, a quadrature, and $(m-1)$ differentiations. Since u and v are given, Arts. 62-68, for the G_1 of those Articles, of course the corresponding invariant differential equations of the m^{th} order may be found, as indicated above, by mere differentiations.

In accordance with the definitions of differential invariants of the first and second orders, given Arts. 57 and 106, we now define the function

$$\frac{d^iv}{du^i}$$

as *a differential invariant of the* $(i+1)^{th}$ *order* of the G_1 Uf.

124. By Arts. 42 and 43 it is clear that the necessary and sufficient condition that a given differential equation of the m^{th} order,

$$\Omega(x, y, y', \dots y^{(m)})=0,$$

shall be invariant under a given G_1 Uf, is that the expression

$$U^{(m)}(\Omega)\equiv\xi\frac{\partial\Omega}{\partial x}+\eta\frac{\partial\Omega}{\partial y}+\eta'\frac{\partial\Omega}{\partial y'}+\dots+\eta^{(m)}\frac{\partial\Omega}{\partial y^m}$$

shall be zero, either identically, or by means of $\Omega=0$,—where

$$U^{(m)}f\equiv\xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y}+\eta'\frac{\partial f}{\partial y'}+\dots+\eta^{(m)}\frac{\partial f}{\partial y^m}$$

is the m-times extended G_1.

In order to reduce the problem of integrating $\Omega=0$, we must first find u, by integrating, if necessary,

$$\frac{dx}{\xi}=\frac{dy}{\eta}.$$

Then v may be found by a quadrature; and the above differential equation of the m^{th} order may be put into the form

$$\mathrm{F}\left(u, v, \frac{dv}{du}, \dots \frac{d^{m-1}v}{du^{m-1}}\right)=0.$$

or, if we choose,

$$\frac{d^{m-1}v}{du^{m-1}}-\Phi\left(u, v, \frac{dv}{du}, \dots \frac{d^{m-2}v}{du^{m-2}}\right)=0.$$

This is a differential equation of the $(m-1)^{\text{th}}$ order in u and v. If its general integral has been found in the form

$$v-f(u, c_1, c_2, \dots c_{m-1})=0,$$

the last equation will be a differential equation of the first order in x and y, which, by Art. 42, must admit of the given G_1 Uf; and hence the general integral of $\Omega=0$ may be finally found by a quadrature.

SECTION II.

Classes of Lie's Differential Equations of the m^{th} Order.

125. We shall now illustrate the method given in the last section for finding classes of invariant differential equations of the m^{th} order by some of the simplest possible examples.

126. *To find all differential equations of the m^{th} order which are invariant under the G_1 of translations along the x-axis,*

$$Uf \equiv \frac{\partial f}{\partial x}.$$

By Art. 62, we have

$$u \equiv y,\ v \equiv y',\ w \equiv \frac{dv}{du} \equiv \frac{y''}{y'},$$

$$\frac{dw}{du} \equiv \frac{d^2v}{du^2} \equiv \frac{y'''y' - y''^2}{y'^3},\ \text{etc.}$$

Thus the most general invariant equation of the m^{th} order may obviously be written

$$\mathrm{F}\left(y,\ y',\ \frac{y''}{y'},\ \frac{y'''y' - y''^2}{y'^3},\ \dots\right) = 0;$$

or, in the equivalent form,

$$\Omega(y,\ y',\ y'',\ y''',\ \dots\ y^{(m)}) = 0.$$

Hence, an equation of the m^{th} order, $\Omega = 0$, which is free of x, admits of the G_1, Uf; and may always be written as an equation of the $(m-1)^{\text{th}}$ order in the variables $u \equiv y$, $v \equiv y'$, in the form

$$\mathrm{F}\left(u,\ v,\ \frac{dv}{du},\ \cdots\ \frac{d^{m-1}v}{du^{m-1}}\right) = 0.$$

127. *To find all differential equations of the m^{th} order which are invariant under the G_1 of all translations along the y-axis,*

$$Uf \equiv \frac{\partial f}{\partial y}.$$

Here, by Art. 62, we have

$$u \equiv x, \quad v \equiv y'; \quad w \equiv \frac{dv}{du} \equiv y'',$$

$$\frac{dw}{du} \equiv \frac{d^2v}{du^2} \equiv y''', \text{ etc.}$$

Hence, in this case, the most general invariant differential equation of the m^{th} order has the form

$$\mathrm{F}(x, y', y'', \dots y^{(m)}) = 0;$$

and it is obvious that it may be written as an equation of the $(m-1)^{\text{th}}$ order in the form

$$\mathrm{F}\left(u, v, \frac{dv}{du}, \dots \frac{d^{m-1}v}{du^{m-1}}\right) = 0.$$

128. *To find all differential equations of the m^{th} order which are invariant under the G_1*

$$Uf \equiv y\frac{\partial f}{\partial y}.$$

Here we have, Art. 63,

$$u = x, \quad v = \frac{y'}{y}; \quad w \equiv \frac{dv}{du} \equiv \frac{y''y - y'^2}{y^2} \equiv \frac{y''}{y} - \left(\frac{y'}{y}\right)^2,$$

$$\frac{dw}{du} \equiv \frac{d^2v}{du^2} \equiv \frac{y'''}{y} - 3\frac{y'}{y}\frac{y''}{y} + 2\left(\frac{y'}{y}\right)^3, \text{ etc.}$$

Hence the most general invariant differential equation of the m^{th} order may be written

$$\mathrm{F}\left(x, \frac{y'}{y}, \frac{y''}{y} - \left(\frac{y'}{y}\right)^2, \frac{y'''}{y} - 3\frac{y'}{y}\frac{y''}{y} + 2\left(\frac{y'}{y}\right)^3, \dots\right) = 0;$$

or in the equivalent form

$$\Omega\left(x, \frac{y'}{y}, \frac{y''}{y}, \frac{y'''}{y}, \dots \frac{y^{(m)}}{y}\right) = 0.$$

Thus every equation of the form $\Omega = 0$, of the m^{th} order in x and y, may be written in the form of an equation of the $(m-1)^{\text{th}}$ order in u and v, by making the substitutions indicated above.

It is seen that the so-called *abridged* linear equation of the m^{th} order, of the form

$$y^{(m)} + X_{m-}{}^{1}(x)y^{(m-1)} + \dots + X_2(x)y' + X_1(x)y = 0,$$

is a particular case of the equation $\Omega = 0$.

129. *To find all differential equations of the m^{th} order which are invariant under the G_1*

$$Uf \equiv \phi(x)\frac{\partial f}{\partial y}.$$

Here, by Art. 115, we have for u and v the forms

$$u \equiv x,\ \ v \equiv \phi y' - \phi' y\,;\ \ \frac{dv}{du} \equiv \phi y'' - \phi'' y,$$

$$\frac{d^2 v}{du^2} \equiv \phi y''' - \phi''' y + \phi' y'' - \phi'' y', \text{ etc.}$$

Hence, the most general invariant differential equation of the m^{th} order may be written

$$\mathrm{F}(x, \phi y' - \phi' y, \phi y'' - \phi'' y, \phi y''' - \phi''' y + \phi' y'' - \phi'' y', \dots) = 0.$$

Any differential equation of the m^{th} order of this form may be written as an equation of the $(m-1)^{\text{th}}$ order in the variables u and v by making the substitutions indicated above.

It may be noticed that the *general linear* equation of the m^{th} order

$$y^{(m} + X_{m-1}(x)y^{(m-1} + \dots + X_2(x)y' + X_1(x)y + X_0(x) = 0$$

is a particular case of the equation $F=0$, if ϕ is an integral-function of the corresponding abridged linear equation. The proof is precisely analogous to that of Art. 115 for equations of the second order.

130. It will be very valuable exercise for the reader to find the differential equations of the 3rd, 4th, ... orders, which are invariant under the simple G_1's given in Art. 117.

131. It may be noticed that the simplest form of differential equation of the m^{th} order that is invariant under the G_1

$$Uf \equiv \frac{\partial f}{\partial y}$$

is

$$y^{(m)} = X(x), \qquad \text{........................}(1)$$

where X is a function of x alone. It is obvious that the general integral of this differential equation of the m^{th} order may be found by m successive quadratures.

132. It is clear that when the equation of the type of Art. 126 has the special form

$$\Omega(y^{(i)}, y^{(i+1)}, \dots y^{(m)}) = 0, \qquad i < m$$

its integration may be facilitated by assuming $y^{(i)} \equiv z$. The equation $\Omega = 0$ is then of only the $(m-i)^{\text{th}}$ order in the variables x and z,

$$\Omega\left(z, \frac{dz}{dx}, \dots \frac{d^{m-i}z}{dx^{m-}}\right) = 0,$$

and is of the type of Art. 126 still.

Its integration will give a result of the form

$$z \equiv y^{(i)} = \Phi(x, c, \dots c_{m-i});$$

so that, by the preceding article, the general integral required may now be found by i successive quadratures.

This method is particularly applicable when $\Omega=0$ has one of the simple forms

$$\Omega \equiv y^{(m)}-f(y^{(m-1)})=0, \text{ ....}(2)$$

or

$$\Omega \equiv y^{(m)}-f(y^{(m-2)})=0. \text{ ....}(3)$$

The first equation, when we put $y^{(m-1)} \equiv z$, becomes

$$\frac{dz}{dx}=f(z),$$

and may be integrated by a quadrature giving z as a function of x and one constant.

The second equation, when the same substitution is made, becomes

$$\frac{d^2z}{dx^2}=f(z),$$

for the integration of which a method is given in Art. 110, by means of which z is found as a function of x and two constants.

Example 1. Given $y''y'''=\sqrt{1+y''^2}$.

This is an example of equation (2), Art. 132. Hence, assume $y'' \equiv z$, and we find

$$z\frac{dz}{dx}=\sqrt{1+z^2},$$

whence $x=c+\sqrt{1+z^2}$. $(c=\text{const.})$

Thus, solving for z, we have

$$z=\sqrt{(x-c)^2-1},$$

or

$$y''=\sqrt{(x-c)^2-1}.$$

By two successive quadratures the general integral may now be found.

Example 2. Given $a^2y^{iv}=y''$.

This is an example of equation (3), Art. 132. If we write z for y'', the equation becomes $a^2z''=z$.

By Art. 110, we find from this

$$z=c_1e^{\frac{x}{a}}+c_2e^{-\frac{x}{a}}.$$

Hence, from $$y''=c_1e^{\frac{x}{a}}+c_2e^{-\frac{x}{a}},$$
by two successive quadratures, we find the general integral

$$y=c_1a^2e^{\frac{x}{a}}+c_2a^2e^{-\frac{x}{a}}+c_3x+c_4.$$

EXAMPLES.

Integrate the following differential equations:

(1) $xy'''=2.$

(2) $ay'''=y'.$

(3) $y^{iv}=\frac{1}{(x+a)^2}.$

(4) $y^{iv}=x\cos x.$

(5) $x^2y^{iv}=2y''.$

(6) $e^xy^{iv}+4\cos x=0.$

(7) $y^{(n)}=xe^x.$

(8) $y'''=\sin^3x.$

(9) $y'''=1+\cos x.$

(10) $y'''=y'(1+y').$

(11) $2xy'''y''=y''^2-a^2.$

(12) $y'''y'^3=1.$

(13) $xy^{iv}+3y'''=0.$

(14) $y'''y''=(1-y''')(1+y''^2)^{\frac{1}{2}}.$

CHAPTER XI.

THE GENERAL LINEAR DIFFERENTIAL EQUATION IN TWO VARIABLES.

133. In the first section of the present chapter we shall give a method for finding, by mere algebraic operations, the general integral of the ordinary linear differential equation of the m^{th} order with constant coefficients and the second member zero.

In the second section we shall give methods for the same equation, when the second member is *not* zero.

In the third section we shall show how the knowledge of the fact that the general linear differential equation is always invariant under a known G_1 may be used to lower the order of the equation.

SECTION I.

The Abridged Linear Equation of the m^{th} Order, with Constant Coefficients.

134. The differential equation of the m^{th} order of the form

$$y^{(m)}+X_{m-1}(x)y^{(m-1)}+\ldots+X_1(x)y=X(x) \text{ ......(1)}$$

is known as *the general linear differential equation* of the m^{th} order. We reserve for the third section a treatment of this equation when the X_i are functions of

x; for the present we assume that the X_i are all constants, so that (1) may be written

$$y^{(m)}+A_{m-1}y^{(m-1)}+\dots+A_1y=X(x). \quad \dots\dots\dots(2)$$

The last equation is known as the general linear equation of the m^{th} order *with constant coefficients.* If, in particular,

$$X(x)\equiv 0,$$

the equation (2) becomes

$$y^{(m)}+A_{m-1}y^{(m-1)}+\dots+A_2y'+A_1y=0, \quad \dots\dots(3)$$

which is called the *abridged* linear equation corresponding to (2). We shall discuss equation (3) in this section.

135. Although, as we know from Art. 128, (3) is invariant under the G_1

$$Uf\equiv y\frac{\partial f}{\partial y},$$

so that the integration of (3) may be reduced by the method of that article, a more expeditious way of finding the general integral, and one not involving any processes of integration, will now be explained.

To this end, substitute in (3) for y the value

$$y\equiv e^{ax},$$

where a is a constant to be determined. It is seen that each of the terms of (3) will be multiplied by the factor e^{ax}, which may therefore be discarded, so that we have

$$a^m+A_{m-1}a^{m-1}+\dots+A_2a+A_1=0. \quad \dots\dots\dots(4)$$

This is an algebraic equation of the m^{th} degree in terms of a; and for each root, a_i, of this equation, it is clear that we have a corresponding particular integral of (3) of the form

$$y_i=e^{a_ix}.$$

Thus, if $a_1, a_2, \dots a_m$ be the m roots of (4), the equation

$$y=c_1e^{a_1x}+c_2e^{a_2x}+\dots+c_me^{a_mx}, \quad \dots\dots\dots\dots(5)$$

will be the general integral of (3), Art. 122. For this equation contains m independent arbitrary constants, and the value of y given by (5), when substituted in (3), satisfies (3) identically.

Example. Given the abridged linear equation of the second order,

$$y''-5y'+6y=0.$$

The corresponding algebraic equation of the second degree is

$$a^2-5a+6=0$$

with the roots 2 and 3. The general integral of the differential equation is therefore, by (5),

$$y=c_1e^{2x}+c_2e^{3x}.$$

That this is correct may be immediately verified.

136. In the case when (4) has a double root, say

$$a_1\equiv a_2,$$

the equation (5) no longer represents the general integral of (3). For in that case the first two terms in (5) reduce to the form

$$(c_1+c_2)\,.\,e^{a_1x},$$

where c_1+c_2 may obviously be replaced by a single arbitrary constant c; and since (5) now only contains $(m-1)$ independent arbitrary constants, it is no longer the general integral of (3).

In order to obtain the general integral, let us suppose that the above-mentioned two roots are not exactly equal, but that they differ by a quantity κ, which will ultimately be made to vanish. The part of (5) depending upon the roots a_1 and a_2 will then have the form

$$c_1e^{a_1x}+c_2e^{(a_1+\kappa)x} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

$$\equiv e^{a_1x}\left\{c_1+c_2\left(1+\kappa x+\frac{\kappa^2x^2}{\lfloor\underline{2}}+\ldots\right)\right\}$$

$$\equiv e^{a_1x}\left\{(c_1+c_2)+c_2\kappa x+c_2\kappa\frac{\kappa x^2}{\lfloor\underline{2}}+\ldots\right\}.$$

Since c_1 and c_2 are arbitrary, we may assume them to be infinite in such manner that, as κ approaches zero, $C_2\kappa$ approaches a finite quantity B_2, while c_1 and c_2 are taken with opposite signs, in such manner that c_1+c_2 is finite and equal to B_1. Thus the sum (6) has the form

$$e^{a_1x}\left\{B_1+B_2x+B_2\frac{\kappa x^2}{\underline{|2}}+\dots\right\},$$

so that, when $\kappa=0$, (6) becomes

$$e^{ax}\{B_1+B_2x\}.$$

Thus we see that in the case when (4) has a double root $a_1\equiv a_2$, the arbitrary constant (C_1+C_2) must be replaced in (5) by a binomial expression of the form

$$(B_1+B_2x).$$

In an entirely analogous manner it may be shown that if (4) has an r-fold root, the r terms coalescing in (5) must be replaced by a polynomial of the form

$$e^{a_ix}\{B_1+B_2x+\dots+B_rx^{r-1}\}.$$

Example. The algebraic equation corresponding to

$$y''-6y'+9y=0$$

has the double root 3. Hence the general integral of the differential equation is

$$y=e^{3x}(B_1+B_2x).$$

137. When (4) has a pair of *imaginary* roots, the corresponding constants of integration are to be assumed imaginary in order that the pair of terms in (5) may be reduced to a real form. Thus, if

$$a_1\equiv \alpha+\beta\sqrt{-1},\quad a_2\equiv \alpha-\beta\sqrt{-1}$$

be a pair of imaginary roots, the corresponding terms in (5) are

$$c_1\,.\,e^{(\alpha+\beta\sqrt{-1})x}+c_2\,.\,e^{(\alpha-\beta\sqrt{-1})x}\equiv e^{\alpha x}(c_1\,.\,e^{\beta\sqrt{-1}\,.\,x}+c_2\,.\,e^{-\beta\sqrt{-1}\,.\,x})$$

$$\equiv e^{\alpha x}[(c_1+c_2)\cos\beta x+\sqrt{-1}\,.\,(c_1-c_2)\sin\beta x].$$

c_1 and c_2 be considered imaginary, and if we

$$c_1+c_2=A,\quad (c_1-c_2)\sqrt{-1}=B,$$

l form sought will be

$$e^{\alpha x}(A\cos\beta x+B\sin\beta x). \quad\ldots\ldots\ldots\ldots(7)$$

is readily seen, as in Art. 136, that if a pair of r-fold
ginary roots occurs in (4), each of the arbitrary con-
nts in (7) must be replaced by a polynomial of the
$-1)^{\text{th}}$ degree of the form

$$B_1+B_2x+\ldots+B_rx^{r-1}.$$

Example. Given $y''-6y'+13y=0.$

The corresponding algebraic equation,

$$a^2-6a+13=0,$$

has the pair of imaginary roots,

$$a_1=3+2\sqrt{-1},\quad a_2=3-2\sqrt{-1}.$$

Hence, by (7), the general integral is seen to be

$$y=e^{3x}(A\cos 2x+B\sin 2x).$$

SECTION II.

The Linear Equation of the m^{th} Order, with Constant Coefficients and the Second Member a Function of x.

138. The problem of finding the general integral of the equation

$$y^{(m)}+A_{m-1}y^{(m-1)}+\ldots+A_1y=X(x)\quad\ldots\ldots\ldots\ldots(1)$$

is intimately connected with that of finding the general integral of the corresponding abridged equation

$$y^{(m)}+A_{m-1}y^{(m-1)}+\ldots+A_1y=0.\quad\ldots\ldots\ldots\ldots(2)$$

For suppose that the general integral of (2) has been found in the form

$$y=c_1e^{a_1x}+c_2e^{a_2x}+\ldots+c_me^{a_mx},\quad\ldots\ldots\ldots\ldots(3)$$

and that $\phi(x)$ is any function of x which satisfies (1),—then it is clear that

$$y = c_1 e^{a_1 x} + c_2 e^{a_2 x} + \ldots + c_m e^{a_m x} + \phi(x) \text{........(4)}$$

is the general integral of (1). For, if we denote the second member of (3) by Y, the result of substituting

$$y = Y + \phi(x) \text{ ....(4)}$$

in (1) will be equal to the sum of the results of substituting

$$y = Y, \quad y = \phi(x)$$

successively. The first of these results will be zero, because Y satisfies equation (2); the second result will be $X(x)$, because $\phi(x)$ satisfies (1). Hence (4), which contains m independent arbitrary constants, must, Art. 122, be the general integral of (1).

We shall call the function $\phi(x)$, which does not contain any arbitrary constant, a *particular integral-function* of (1); the function Y is usually called the *complementary* function.

Thus, when a particular integral-function of (1) is known, the general integral of (1) may be found by adding this function to the general integral of (2).

139. There are numerous methods for finding the general integral (4); and probably the most elegant is that based upon the theory of Transformation Groups. But this method will have to be reserved for a later occasion, since its application presupposes a knowledge of groups of more than one parameter.

The method which most naturally suggests itself is, by differentiation and (if necessary) elimination, to derive from (1) a differential equation of the $(m+n)^{\text{th}}$ order in which the second member is *zero*. The general integral of this equation, found by Art. 135, will contain $(m+n)$ arbitrary constants, and will be of the form

$$y = B_1 e^{b_1 x} + B_2 e^{b_2 x} + \ldots + B_m e^{b_m x} + \ldots + B_{m+n} e^{b_{m+n} x}, \text{...(5)}$$

where the B_i are arbitrary constants. But this general integral of the differential equation of the $(m+n)^{\text{th}}$ order must necessarily *include* the general integral of (1); and since the complementary function of (1) has the form (3), m of the b_i above must coincide with the m a_i in (3). Thus the algebraic equation of the $(m+n)^{\text{th}}$ degree of which the b_i are the roots, Art. 135, has m roots in common with the algebraic equation corresponding to the abridged equation (2). Hence, when the a_i have been found, the n b_κ in (5), which do not coincide with the a_i, may be found by solving an algebraic equation of the n^{th} degree only.

Suppose now that $b_1 = a_1, \dots b_m = a_m$; then the n arbitrary constants $B_{m+1}, \dots B_{m+n}$ in (5) are superfluous for the general integral of (1); and these n constants may be so determined, by substituting the value of y from (5) in (1), that (1) will be satisfied. Since the value of y given by (5) must satisfy (1), no matter what values the $B_1, \dots B_m$ have, we may facilitate the determination of the $B_{m+1}, \dots B_{m+n}$ by assuming $B_1 = 0, \dots B_m = 0$.

Example 1. Given $y'' - n^2y = x + 1.$ ....(6)

By differentiating twice we find

$$y^{\text{iv}} - n^2y'' = 0; \qquad (7)$$

and the algebraic equation corresponding to (7) is

$$a^4 - n^2a^2 = 0. \qquad (8)$$

The algebraic equation corresponding to the abridged equation

$$y'' - n^2y = 0$$

is

$$a^2 - n^2 = 0,$$

of which the roots are $a = n$, $a = -n$. Hence—as in this case is evident anyhow, $a = n$ and $a = -n$ are two roots of the equation of the fourth degree (8). The other roots are obviously $a = 0$ repeated twice. Hence, Art. 136, the general integral of (7) is

$$y = B_1e^{nx} + B_2e^{-nx} + B_3 + B_4x,$$

while the complementary function of (6) is

$$C_1e^{nx} + C_2e^{-nx}.$$

Thus we must be able to determine B_3 and B_4 in such manner that

$$B_3+B_4x$$

will be a particular integral-function of (6). Substituting

$$y=B_3+B_4x$$

in (6), we find

$$-n^2(B_3+B_4x)\equiv 1+x\,;$$

and hence

$$B_3\equiv B_4\equiv -\frac{1}{n^2}.$$

Thus the general integral of (6) is

$$y=c_1e^{nx}+c_2e^{-nx}-\frac{1+x}{n^2}.$$

Example 2. Given

$$y''+8y'+16y=4e^x-e^{2x}. \qquad\text{.........................(9)}$$

By differentiating twice we may eliminate e^x and e^{2x}, and find

$$y^{iv}+5y'''-6y''-32y'+32y=0.$$

The corresponding algebraic equation is

$$a^4+5a^3-6a^2-32a+32=0\,; \qquad\text{..................(10)}$$

and this equation has two roots in common with

$$a^2+8a+16=0,$$

of which the roots are -4 and -4. Hence the remaining two roots of the algebraic equation of the fourth degree may be determined from the equation which results when (10) has been divided by the factor $(a+4)^2$, that is, from

$$a^2-3a+2=0.$$

The other two roots of (10) are therefore $a=1$, $a=2$.

The general integral of (9) has therefore the form

$$y=(B_1+B_2x)e^{-4x}+B_3e^x+B_4e^{2x},$$

so that it must be possible to determine B_3 and B_4 in such manner that

$$B_3e^x+B_4e^{2x}$$

is a particular integral-function of (9).

By substituting $\qquad y=B_3x+B_4e^{2x}$

in (9), we find

$$B_3e^x+4B_4e^{2x}+8B_3e^x+16B_4e^{2x}+16B_3e^x+16B_4e^{2x}\equiv 4e^x-e^{2x}\,;$$

that is,

$$B_3\equiv\frac{4}{25},\ B_4\equiv -\frac{1}{36}.$$

Thus we find for (9) the general integral

$$y=(c_1+c_{2x})e^{-4x}+\frac{4}{25}e^x-\frac{1}{36}e^{2x}.$$

140. A second method for finding a particular integral-function of (1) is that which is commonly known as the "Variation of Parameters." To find $\phi(x)$ (Art. 138), by this method, we first find the general integral of (2) in the form (3); then, considering the m arbitrary constants as variable parameters, by substituting the value of y given by (3) in (1), we determine the parameters in such manner that (1) is satisfied.

The m parameters may evidently be subjected to $(m-1)$ arbitrary conditions; and the system of conditions which produces the simplest result is that which demands that all the derivatives of y of an order lower than the m^{th} shall have the *same* values when the parameters are considered as variables that they have when the parameters are considered as constants.

Example. Given $$y''+n^2y=X(x). \quad\text{.........................}(11)$$
The general integral of the abridged equation corresponding to (11) is
$$y=c_1\cos nx+c_2\sin nx. \quad\text{.......................}(12)$$

Now, supposing c_1 and c_2 to be variables, we wish to determine these quantities in the simplest manner possible, so that (12) will be the general integral of (11). Differentiating (12), we have

$$y'=-nc_1\sin nx+nc_2\cos nx+\cos nx\frac{dc_1}{dx}+\sin nx\frac{dc_2}{dx};$$

and thus, in order that y' may have the same value as if c_1 and c_2 were constants, we must have

$$\cos nx\frac{dc_1}{dx}+\sin nx\frac{dc_2}{dx}=0. \quad\text{....................}(13)$$

Also, differentiating the equation

$$y'=-nc_1\sin nx+nc_2\cos nx$$

again, we find

$$y''=-n^2(c_1\cos nx+c_2\sin nx)-n\sin nx\frac{dc_1}{dx}+n\cos nx\frac{dc_2}{dx}.$$

In order that y shall satisfy (11) we must have, therefore,

$$-n\sin nx\frac{dc_1}{dx}+n\cos nx\frac{dc_2}{dx}=X(x); \quad\text{..............}(14)$$

and from (13) and (14) we find

$$-n\frac{dc_1}{dx}=X\sin nx,\quad n\frac{dc_2}{dx}=X\cos nx.$$

Hence, by two quadratures,

$$c_1=-\frac{1}{n}\int X\sin nx\,dx+a_1,\quad c_2=\frac{1}{n}\int X\cos nx\,dx+a_2\,;$$

and the general integral of (11) is

$$y=-\frac{1}{n}\cos nx\int X\sin nx\,dx+\frac{1}{n}\sin nx\int X\cos nx\,dx+a_1\cos nx+a_2\sin nx.$$

It will be seen that the same result may be obtained directly by Art. 146.

141. It should be noticed that all equations of the form

$$(a+bx)^m y^{(m)}+A_1(a+bx)^{m-1}y^{(m-1)}+\dots$$
$$+A_{m-1}(a+bx)y'+A_m y+X(x)=0 \quad \dots\dots(15)$$

may be transformed into linear equations with constant coefficients by the simple substitution

$$a+bx=e^t,$$

t being the new independent variable.

If, in equation (15), the constant a happens to be zero, (15) is called the *general homogeneous linear equation* of the m^{th} order.

Example. The equation

$$(a+bx)^2y''+A_1(a+bx)y'+A_2y=0,$$

when we assume

$$a+bx=e^t,$$

becomes linear with constant coefficients. For we have

$$y'\equiv\frac{dy}{dx}=be^{-t}\frac{dy}{dt},$$

$$y''\equiv\frac{d^2y}{dx^2}=b^2e^{-2t}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right);$$

so that the above equation becomes

$$b^2\frac{d^2y}{dt^2}-(b^2-A_1b)\frac{dy}{dt}+A_2y=0,$$

an equation of the form (2).

SECTION III.

The General Linear Equation of the m^{th} Order in which the Coefficients are Functions of x.

142. It was shown, Art. 129, that the linear equation of the m^{th} order

$$y^{(m)}+X_{m-1}(x)y^{(m-1)}+\ldots+X_1(x)y+X_0(x)=0, \ldots(1)$$

where the X_i are functions of x alone, is invariant under the G_1

$$Uf\equiv\phi(x)\frac{\partial f}{\partial y},$$

if ϕ is any function of x which satisfies the *abridged* linear equation corresponding to (1)

$$y^{(m)}+X_{m-1}(x)y^{(m-1)}+\ldots+X_1(x)y=0.\ldots\ldots\ldots(2)$$

We shall call $\phi(x)$, under these circumstances, a *particular integral-function* of (2).

143. In order to depress the order of equation (1), we know, Art. 129, that we must make the substitutions

$$x\equiv u,\quad y'\equiv\frac{v}{\phi}+\frac{\phi'}{\phi}y,\quad y''\equiv\frac{1}{\phi}\frac{dv}{du}+\frac{\phi''}{\phi}y,$$

$$y'''\equiv\frac{1}{\phi}\frac{d^2v}{du^2}-\frac{\phi'}{\phi^2}\frac{dv}{du}+\frac{\phi''}{\phi^2}v+\frac{\phi'''}{\phi}y,\text{ etc.}$$

Now all the terms in (1) which will contain y explicitly when the above substitutions are made, are evidently included in the expression

$$y\{\phi^m+X_{m-1}(u)\phi^{(m-1)}+\ldots+X_1(u)\phi\};$$

and, since ϕ is a particular integral-function of (2), the above expression is identically zero. That is, the equation of the $(m-1)^{\text{th}}$ order, which corresponds to the equation (1) in the variables u and v, is also *linear*. Hence, *if a particular integral-function, $\phi(x)$, of the abridged linear* *equation* (2) *is known, the general linear equation* (1) *will admit of the* G_1

$$Uf \equiv \phi(x)\frac{\partial f}{\partial y},$$

and will be linear, of the $(m-1)^{th}$ *order, when written in the variables* u *and* v *by the method of Art.* 129. *Thus the complete integration of* (1) *may, in this case, be accomplished by the integration of a linear differential equation of the* $(m-1)^{th}$ *order and a quadrature.*

144. If, in particular, $m=2$ in equation (1), we see that *the general linear differential equation of the second order*

$$y''+X_2(x)y'+X_1(x)y+X_0(x)=0 \text{ ....}(1')$$

may be completely integrated by two quadratures, when a particular integral-function, $\phi(x)$, *of the corresponding abridged equation*

$$y''+X_2(x)y'+X_1(x)y=0 \text{..................}(2')$$

is known.

Written in the variables u and v, Art. 129, (1′) becomes

$$\frac{dv}{du}+X_2v+X_0\phi+\{\phi''+X_2\phi'+X_1\phi\}y=0;$$

or, since ϕ satisfies (2′),

$$\frac{dv}{du}+X_2v+X_0\phi=0,$$

X_2, X_0, and ϕ being expressed as functions of u.

The general integral of this linear equation of the first order is, Art. 68,

$$v=e^{-\int X_2(u)du}\left\{-\int X_0(u)\phi(u)e^{\int X_2(u)du}.\,du+c_1\right\};$$

or, restoring the variables x and y,

$$\phi y'-\phi' y=e^{-\int X_2(x)dx}\left\{-\int X_0(x)\phi(x)e^{\int X_2(x)dx}.\,dx+c_1\right\}.$$

The general integral of *this* linear equation of the first order, that is, the general integral of (1′) is, Art. 68,

$$y=\phi(x)\left\{\int e^{-\int X_2(x)dx}\left[-\int X_0(x)\phi(x)e^{\int X_2(x)dx}dx+c_1\right]\frac{dx}{\phi^2}+c_2\right\}.$$

145. The abridged linear differential equation of the m^{th} order (2) admits of the G_1

$$Uf\equiv y\frac{\partial f}{\partial y},$$

so that the order of (2) may always be depressed by unity by the method of Art. 128; but the resulting differential equation of the $(m-1)^{\text{th}}$ order in the variables u and v is usually *not* linear.

146. If, in particular, we assume that (2) has constant coefficients, and is only of the second order, of the form

$$y''+Ay'+By=0, \qquad (A, B \text{ const.})$$

written in u and v, by Art. 128, it becomes

$$\frac{dv}{du}+v^2+Av+B=0;$$

an equation of the first order which may obviously be integrated by a quadrature, when another quadrature will give the general integral of the differential equation of the second order. In this manner the particular integral-function $\phi(x)$ may be found,—that is, by ascribing any numerical values desired to the two arbitrary constants in the value of y found. Also, $\phi(x)$ may be found by Art. 135.

When $\phi(x)$ is known, the general linear equation of the second order, with constant coefficients

$$y''+Ay'+By+X(x)=0,$$

may, by Art. 129, be written as a *linear* differential equation of the first order in u and v. The last equation may be integrated by a quadrature, Art. 68; when a second quadrature will give the general integral of the general linear equation of the second order with constant coefficients.

Hence, since $\phi(x)$ may always be found, by Art. 135, by algebraic operations, we see that *the linear differential equation of the second order with constant coefficients may always be integrated by two quadratures.*

For practical work, however, the method of Art. 139 will usually be found more advantageous.

Example. Given the differential equation

$$y''+n^2y=\cos nx. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

It is seen that $\sin nx$ is a particular integral-function of the abridged equation

$$y''+n^2y=0.$$

Hence the above differential equation of the second order admits of

$$Uf\equiv\sin nx\frac{\partial f}{\partial y};$$

and to depress the order of the equation we have, Art. 129, to substitute in (3),

$$x\equiv u,\ y'\equiv\frac{v}{\sin nx}+n\cot nxy,\ \ y''=\frac{1}{\sin nx}\frac{dv}{du}-n^2y.$$

We find

$$\frac{1}{\sin nu}\frac{dv}{du}=\cos nu,$$

or

$$dv=\sin nu\cos nu\,du.$$

Hence

$$v=\frac{\sin^2 nu}{2n}+c_1, \qquad (c_1=\text{const.})$$

or, since

$$v\equiv\sin nxy'-n\cos nxy,\ \ u\equiv x,$$

we have,

$$\sin nxy'-n\cos nxy=\frac{\sin^2 nx}{2n}+c_1.$$

The integral of this linear differential equation of the first order may, by Art. 68, be found by a quadrature in the form

$$y=\frac{x\sin nx}{2n}-\frac{c_1\cos nx}{n}+c_2\sin nx.$$

EXAMPLES.

Integrate the following abridged linear equations with constant coefficients :

(1) $y'' - 7y' + 12y = 0.$

(2) $3y'' - 10y' + 3y = 0.$

(3) $y''' - 4y' = 0.$

(4) $y''' - 7y' + 6y = 0.$

(5) $y^{iv} - 12y'' + 27y = 0.$

(6) $y^{iv} - 4y''' + 6y'' - 4y' + y = 0.$

(7) $y'' - 4aby' + (a^2 + b^2)^2 y = 0.$

(8) $y''' + y'' + y' - 3y = 0.$

(9) $y^{iv} + 2y'' - 8y = 0.$

(10) $y''' - 3y'' + 4y = 0.$

(11) $y^{iv} + 2n^2 y'' + n^4 y = 0.$

(12) $y^{iv} - 3y''' + 3y'' - y' = 0.$

Integrate the following linear equations with constant coefficients and the second members functions of x :

(13) $y'' - 7y' + 12y = x.$

(14) $y^{iv} - 2y''' + 2y'' - 2y' + y = 1.$

(15) $y''' - 2y'' + y' = e^x.$

(16) $y'' + n^2 y = 1 + x + x^2.$

(17) $y'' - 2y' + y = e^x.$

(18) $y'' - 3y' + 2y = xe^{nx}.$

(19) $y'' + 4y = x \sin^2 x.$

(20) $y''' - 2y' + 4y = e^x \cos x.$

Integrate the following equations by the method of Art. 141 :

(21) $x^2 y'' - xy' - 3y = 0.$

(22) $(x+a)^2 y'' - 4(x+a)y' + 6y = 0.$

(23) $x^2 y'' - xy' + 2y = x \log x.$

(24) $(2x-1)^3 y''' + (2x-1)y' - 2y = 0.$

Integrate the following equations by the method of Sec. III. :

(25) $(1-x^2)y'' + xy' - y = x(1-x^2)^{\frac{3}{2}}.$

(26) $y'' - xy' + (x-1)y = x^2.$

(27) $x^2 y'' + 4xy' + 2y = e^x.$

(28) $xy'' + 2y' = x.$

(29) $(1-x^2)y'' - xy' - y = 0.$

(30) $y'' - \dfrac{x}{x-1}y' + \dfrac{1}{x-1}y = x - 1.$

Other examples for practice may be found in the Examples at the ends of Chapters IX. and X.

CHAPTER XII.

METHODS FOR THE INTEGRATION OF THE SIMULTANEOUS SYSTEM.

147. In the first section of this chapter we shall give briefly the simplest of the methods which do not involve transformation groups for integrating certain forms of simultaneous systems of ordinary differential equations.

In the second section we shall give a general method of integration for a simultaneous system in three variables, when the equivalent linear partial differential equation of the first order in three variables admits of a known G_1; while in the third section we shall give an application of the theory developed in the second section to ordinary differential equations of the second order in two variables.

SECTION I.

Special Methods for Integrating Certain Forms of Simultaneous Systems.

148. In Art. 23, Chap. II., we gave a method for integrating a simultaneous system of the form

$$\frac{dx}{X(x, y)} = \frac{dy}{Y(x, y)} = \frac{dz}{Z(x, y, z)};$$

that is, we saw that when the first equation had been integrated,—by the methods of Chapter IV.,—either x or

y might be eliminated from Z, so that a second integral of the simultaneous system might be found by integrating a second differential equation of the first order in two variables. It is obvious, therefore, that a simultaneous system of the above form may be completely integrated by integrating two ordinary differential equations of the first order in two variables.

149. The general form of the simultaneous system in three variables is

$$\frac{dx}{X}=\frac{dy}{Y}=\frac{dz}{Z}, \quad \text{.......................}(1)$$

where X, Y, and Z are usually functions of all three variables x, y, z.

We may write the ratios (1)

$$\frac{dx}{X}=\frac{dy}{Y}=\frac{dz}{Z}=\frac{\lambda dx+\mu dy+\nu dz}{\lambda X+\mu Y+\nu Z}, \quad \text{............}(2)$$

where λ, μ, ν may be either constants or functions of the variables. If it is possible to choose λ, μ, ν in such manner that

$$\lambda X+\mu Y+\nu Z=0,$$

then also

$$\lambda dx+\mu dy+\nu dz=0; \quad \text{................}(3)$$

and the integral-function of the total differential equation (3), if it may be found by the methods of Chap. VIII., will obviously also be one of the integral-functions of (1). That is, if

$$\Omega(x, y, z)=c$$

is the integral of (3), it is also an integral of (1).

The second integral of (1) may then be found by integrating an ordinary differential equation in two variables from, say,

$$\frac{dx}{X}=\frac{dy}{Y},$$

when z has been eliminated from X and Y by means of $\Omega=c$.

Example. Given the simultaneous system

$$\frac{dx}{mx-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx}.$$

The method of the present article may be applied twice. If we choose λ, μ, ν equal to l, m, n respectively, we find

$$ldx+mdy+ndz=0.$$

If we choose λ, μ, ν equal to x, y, z respectively, we find

$$xdx+ydy+zdz=0.$$

The integrals of these equations are obviously

$$lx+my+nz=c_1,$$

$$x^2+y^2+z^2=c_2;$$

and these two equations are the general integrals of the given system. For the geometrical meaning of the integrals of a simultaneous system in three variables, see Art. 19.

150. The general simultaneous system in the $(n+1)$ variables $x_1, \ldots, x_n, t$, has, as we know, the form

$$\frac{dx_1}{X_1}=\frac{dx_2}{X_2}=\ldots=\frac{dx_n}{X_n}=\frac{dt}{T}, \quad \ldots\ldots\ldots\ldots\ldots(4)$$

where the $X_1, \ldots, X_n, T$, are usually functions of all the variables. If we choose any one of the variables, say t, as the independent variable, it will always be possible, by differentiating these equations a sufficient number of times, to eliminate all but one of the dependent variables and their differential coefficients. In fact, if no method for abbreviating the work suggests itself, we may always obtain, by differentiating each of the given equations $(n-1)$ times, exactly n^2 equations, which are just sufficient to eliminate $(n-1)$ variables with their $n(n-1)$ differential coefficients. The resulting differential equation of the n^{th} order in two variables must then be integrated; and from its general integral, and the system (4), the values of the other dependent variables may be found, giving a system of general integrals consisting, Art. 20, of n equations involving n arbitrary constants.

Of course this method is most appropriate for the integration of systems of *linear equations with constant coefficients*, since we then have a definite method, Art. 135, for the integration of the system.

151. As an illustration of the preceding article, suppose that a system of *two* differential equations of the first order is given, connecting the variables x, y, and t, t being chosen as the independent variable. To find the equation connecting x and t, we differentiate, if necessary, *both* of the given equations with respect to t; thus obtaining four equations connecting the quantities

$$x,\ y,\ \frac{dx}{dt},\ \frac{dy}{dt},\ \frac{d^2x}{dt^2},\ \frac{d^2y}{dt^2},$$

from which we can eliminate $y, \frac{dy}{dt}, \frac{d^2y}{dt^2}$. The resulting equation will, of course, be a differential equation of the second order in x and t.

The general integral of this equation will give x in terms of t and two arbitrary constants; and by substituting this value of x in one of the equations of the given system, y may be found.

Example 1. Given the simultaneous system of linear equations,

$$\frac{dx}{3x-y}=\frac{dy}{x+y}=\frac{dt}{1}. \quad\text{..............................(5)}$$

These equations may be written

$$\frac{dx}{dt}-3x+y=0,\ \frac{dy}{dt}-x-y=0\,;$$

and by differentiating the first we find,

$$\frac{d^2x}{dt^2}-3\frac{dx}{dt}+\frac{dy}{dt}=0.$$

Hence

$$\frac{d^2x}{dt^2}-3\frac{dx}{dt}+x+y=0\,;$$

or, from the first equation,

$$\frac{d^2x}{dt^2}-4\frac{dx}{dt}+4x=0.$$

The general integral of this linear differential equation of the second order, with constant coefficients, is found by Art. 136 to be

$$x=(B_1+B_2t)e^{2t}. \quad\text{..............................}(6)$$

Substituting this value of x in

$$\frac{dx}{dt}-3x+y=0,$$

we find for y,
$$y=(B_1-B_2+B_2t)e^{2t}. \quad\text{..........................}(7)$$

Thus the equations (6) and (7) represent the system of general integrals of (5).

Example 2. Given the system of linear equations,

$$\frac{dx}{dt}+5x+y=e^t,$$

$$\frac{dy}{dt}-x+3y=e^{2t}, \text{..............................}(8)$$

in which the independent variable t occurs explicitly. Differentiating the first equation, we have

$$\frac{d^2x}{dt^2}+5\frac{dx}{dt}+\frac{dy}{dt}=e^t.$$

By means of the equations (8) we may eliminate y and $\dfrac{dy}{dt}$ from the last equation, giving

$$\frac{d^2x}{dt^2}+8\frac{dx}{dt}+16x=4e^t-e^{2t}.$$

By the method of Art. 139 the general integral of this equation is found to be

$$x=(c_1+c_2t)e^{-4t}+\frac{4}{25}e^t-\frac{1}{36}e^{2t};$$

and this value being substituted into the first of the equations (8) gives us at once,

$$y=-(c_1+c_2+c_2t)e^{-4t}+\frac{7}{36}e^{2t}+\frac{1}{25}e^t.$$

152. It is clear that the differential equation of the second order,

$$y''-\omega(x, y, y')=0,$$

may be regarded as equivalent to a simultaneous system of equations of the first order in three variables. For we have

$$y'=\frac{dy}{dx},\ y''=\frac{dy'}{dx}=\omega(x, y, y'),$$

so that

$$\frac{dx}{1}=\frac{dy}{y'}=\frac{dy'}{\omega(x, y, y')}.$$

In an analogous manner it is clear that a differential equation of the m^{th} order in two variables is equivalent to a simultaneous system of m differential equations of the first order in $(m+1)$ variables.

Similarly, a *simultaneous system* of differential equations of an order higher than the first may always be written as a simultaneous system of differential equations of the *first* order in the proper number of variables.

For example, if in the simultaneous system of the second order

$$\frac{d^2x}{dt^2}=X,\ \frac{d^2y}{dt^2}=Y,\ \frac{d^2z}{dt^2}=Z,$$

where X, Y, Z are certain functions of x, y, z, t,—we designate by x', y', z' the differential coefficients, with respect to t, of x, y, and z respectively,—the above simultaneous system may obviously be written,

$$\frac{dx}{dt}=x',\ \ \frac{dy}{dt}=y',\ \ \frac{dz}{dt}=z',$$

$$\frac{dx'}{dt}=X,\ \frac{dy'}{dt}=Y,\ \frac{dz'}{dt}=Z.$$

Thus the simultaneous system of equations of the second order in four variables may be replaced by the simultaneous system of equations of the first order in seven variables.

If the six general integrals of this system, involving six arbitrary constants, have been found, Art. 150, the elimination of x', y', z' between these integrals will give the three general integrals, involving six arbitrary constants, of the simultaneous system of equations of the second order.

153. A method of integrating a simultaneous system of linear equations with constant coefficients and of an order higher than the first, analogous to that of Art. 150, will be sufficiently illustrated by the following example:

Example. Given the system

$$\frac{d^2x}{dt^2}=7x+3y, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

$$\frac{d^2y}{dt^2}=2x+6y.$$

By differentiating the first equation twice we find

$$\frac{d^4x}{dt^4}=7\frac{d^2x}{dt^2}+3\frac{d^2y}{dt^2};$$

and from this equation, by means of the equations (9), we find

$$\frac{d^4x}{dt^4}-13\frac{d^2x}{dt^2}+36x=0.$$

The general integral of this equation is, by Art. 136,

$$x=(c_1+c_2t)e^{2t}+(c_3+c_4t)e^{3t};$$

and by substituting this value of x in the first of equations (9), we find

$$y=\frac{2}{3}c_4te^{3t}+\left(\frac{2}{3}c_3+2c_4\right)e^{3t}+\left(\frac{4}{3}c_2-c_1\right)e^{2t}-c_2te^{2t}.$$

SECTION II.

Theory of Integration of a Simultaneous System in Three Variables which is Invariant under a known G_1.

154. It was shown in Chapter II. that the general simultaneous system in three variables, of the form

$$\frac{dx}{X}=\frac{dy}{Y}=\frac{dz}{Z},$$

is equivalent to the linear partial differential equation of the first order in the same variables,

$$X\frac{\partial f}{\partial x}+Y\frac{\partial f}{\partial y}+Z\frac{\partial f}{\partial z}=0; \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and that two independent solutions of the latter were always two independent integral-functions of the former, and *vice versa*.

Thus we may consider the linear partial equation (1) as taking the place of the above simultaneous system; and when we speak of the equation (1) *admitting of*, or being *invariant* under a given G_1—an expression which we shall immediately explain—we may also, if we choose, say that the simultaneous system *admits of*, or is *invariant* under the given G_1. The theory of integration of this section will be developed, therefore, for the linear partial differential equation (1), using that equation as the representative of the corresponding simultaneous system.

155. A G_1 in three variables has the general form

$$Uf\equiv\xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y}+\zeta\frac{\partial f}{\partial z}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where ξ, η, ζ are functions of the variables x, y, z. We say that the linear equation

$$Af\equiv X\frac{\partial f}{\partial x}+Y\frac{\partial f}{\partial y}+Z\frac{\partial f}{\partial z}=0, \ldots\ldots\ldots\ldots\ldots(1)$$

—where X, Y, Z are, of course, certain functions of x, y, z—is *invariant* under, or *admits of* the G_1 Uf when, by means of the G_1 Uf, each solution of (1) is transformed (compare Art. 58) into a solution of (1). Thus, if $\omega_1(x, y, z)$, $\omega_2(x, y, z)$ be two independent solutions of (1), using the customary symbolic method for expressing the fact that the transformation Uf is performed upon the function ω_i, the condition that (1) shall be invariant under Uf is

$$U(\omega_i)\equiv\Omega_i(\omega_1, \omega_2) \quad i=1, 2. \ \ldots\ldots\ldots\ldots(3)$$

This condition for the invariance of (1) can, of course, only be applied when the solutions ω_1, ω_2 are known; but we shall in the next article develop a condition which is practicable when ω_1 and ω_2 are unknown.

156. The expression

$$U(Af)-A(Uf)$$

has a definite meaning: it means, for the first term, *put Af in place of f, in Uf*; and, in the second term, *put Uf in place of f, in Af.* Thus it is seen

$$\begin{aligned} U(Af)-A(Uf) \equiv {} & \xi\frac{\partial}{\partial x}\left(X\frac{\partial f}{\partial x}+Y\frac{\partial f}{\partial y}+Z\frac{\partial f}{\partial z}\right) \\ & +\eta\frac{\partial}{\partial y}\left(X\frac{\partial f}{\partial x}+Y\frac{\partial f}{\partial y}+Z\frac{\partial f}{\partial z}\right)+\zeta\frac{\partial}{\partial z}\left(X\frac{\partial f}{\partial x}+Y\frac{\partial f}{\partial y}+Z\frac{\partial f}{\partial z}\right) \\ & -X\frac{\partial}{\partial x}\left(\xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y}+\zeta\frac{\partial f}{\partial z}\right)-Y\frac{\partial}{\partial y}\left(\xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y}+\zeta\frac{\partial f}{\partial z}\right) \\ & -Z\left(\xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y}+\zeta\frac{\partial f}{\partial z}\right). \end{aligned}$$

If the differentiations here indicated are carried out, the terms involving differential coefficients of f of the second order will cancel out. For instance $\xi X\frac{\partial^2 f}{\partial x^2}$ will occur in the first term with a positive sign, and in the fourth term with a negative sign, etc.

Thus we find the noteworthy symbolic expression

$$\begin{aligned} & U(Af)-A(Uf) \\ & \quad \equiv\left\{\xi\frac{\partial X}{\partial x}+\eta\frac{\partial X}{\partial y}+\zeta\frac{\partial X}{\partial z}-X\frac{\partial \xi}{\partial x}-Y\frac{\partial \xi}{\partial y}-Z\frac{\partial \xi}{\partial z}\right\}\frac{\partial f}{\partial x} \\ & \quad +\left\{\xi\frac{\partial Y}{\partial x}+\eta\frac{\partial Y}{\partial y}+\zeta\frac{\partial Y}{\partial z}-X\frac{\partial \eta}{\partial x}-Y\frac{\partial \eta}{\partial y}-Z\frac{\partial \eta}{\partial z}\right\}\frac{\partial f}{\partial y} \\ & \quad +\left\{\xi\frac{\partial Z}{\partial x}+\eta\frac{\partial Z}{\partial y}+\zeta\frac{\partial Z}{\partial z}-X\frac{\partial \zeta}{\partial x}-Y\frac{\partial \zeta}{\partial y}-Z\frac{\partial \zeta}{\partial z}\right\}\frac{\partial f}{\partial z}. \end{aligned}$$

But when it is remembered that

$$U(X) \equiv \xi\frac{\partial X}{\partial x}+\eta\frac{\partial X}{\partial y}+\zeta\frac{\partial X}{\partial z}, \quad A(\xi) \equiv X\frac{\partial \xi}{\partial x}+Y\frac{\partial \xi}{\partial y}+Z\frac{\partial \xi}{\partial z};$$

and that similar expressions hold for $U(Y)$, $A(\eta)$, etc., the above identity may be written (putting for brevity UX for $U(X)$, etc.,)

$$U(Af)-A(Uf)$$
$$\equiv (UX-A\xi)\frac{\partial f}{\partial x}+(UY-A\eta)\frac{\partial f}{\partial y}+(UZ-A\zeta)\frac{\partial f}{\partial z}. \quad \ldots(4)$$

Now if ω_1, ω_2 be the (unknown) solutions of $Af=0$, we must have

$$A(\omega_1) \equiv A(\omega_2) \equiv 0,$$

Hence also

$$U(A(\omega_1)) \equiv U(A(\omega_2)) \equiv 0.$$

Further, from (3), if $Af=0$ admits of the G_1 Uf,

$$A(U(\omega_i)) \equiv A(\Omega_i(\omega_1, \omega_2)) \equiv \frac{\partial \Omega_i}{\partial \omega_1}.A(\omega_1)+\frac{\partial \Omega_i}{\partial \omega_2}.A(\omega_2),\ i=1, 2.$$

But since ω_1, ω_2 are solutions of $Af=0$, this last expression must be zero identically. Thus the whole expression $U(Af)-A(Uf)$ becomes zero if ω_i is put in place of f, and (4) becomes

$$(UX-A\xi)\frac{\partial \omega_i}{\partial x}+(UY-A\eta)\frac{\partial \omega_i}{\partial y}+(UZ-A\zeta)\frac{\partial \omega_i}{\partial z} \equiv 0,$$
$$i=1, 2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Also we know that

$$X\frac{\partial \omega_i}{\partial x}+Y\frac{\partial \omega_i}{\partial y}+Z\frac{\partial \omega_i}{\partial z} \equiv 0; \quad \ldots\ldots\ldots\ldots\ldots(6)$$

so that from (5) and (6) must follow the identities,

$$\frac{UX-A\xi}{X} \equiv \frac{UY-A\eta}{Y} \equiv \frac{UZ-A\zeta}{Z}. \quad \ldots\ldots\ldots(7)$$

Let the value of the ratios (7) be represented by $\lambda(x, y, z)$; then we may write

$$UX - A\xi \equiv \lambda . X,\ UY - A\eta \equiv \lambda . Y,\ UZ - A\zeta \equiv \lambda . Z.$$

Hence from (4)

$$U(Af) - A(Uf) \equiv \lambda\left(X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial y} + Z\frac{\partial f}{\partial z}\right) \equiv \lambda . Af. \text{...(8)}$$

This then is the condition that a linear partial differential equation of the first order shall admit of a given G_1: and it is clear that the condition may at once be extended to n variables (including $n = 2$). It is customary to write (8) in the brief form

$$(U, A) \equiv \lambda . Af, \text{ ....(9)}$$

where the left-hand member of (9) is merely an abbreviation for the left-hand member of (8).

It is easy to see that the necessary condition (8) is also sufficient. For if ω_i is put in place of f in (8), we obtain

$$A(U(\omega_i)) \equiv 0, \text{ ....(10)}$$

since the other terms in (8) vanish identically. But (10) means that $U(\omega_i)$ is a solution of $Af = 0$; that is, if (8) is a true equation, the solutions ω_i must admit of the transformation Uf—that is, the differential equation $Af = 0$ itself must admit of Uf. Hence, the *necessary* condition (8) is also *sufficient*.

157. It is evident from (9) and (8) that every expression of the form (A, A) or (U, U) is identically zero, and hence the condition (9) that the equation $Af = 0$ shall admit of the G_1 $\rho . Af$, where ρ is an arbitrary multiplier, is satisfied. But this transformation $\rho . Af$, which tells us nothing new concerning the equation $Af = 0$, and which has therefore no value in the problem of integration, is said to be *trivial* with respect to $Af = 0$. This accords with the definition of Art. 60 for trivial transformation in two variables. Such transformations are always to be disregarded in our investigations.

158. We shall now for the moment write Uf equal to zero, and consider the equation $Uf=0$ as a second linear partial differential equation, and we shall show that if the condition (9) exists, that is, if

$$(U, A) \equiv \lambda . Af, \text{ ....}(9)$$

then $Uf=0$ and $Af=0$ may be put into forms for which $(U, A) \equiv 0$, and ultimately that these equations have one solution *in common.*

When the condition (9) holds, and λ is not zero, the two linear partial equations $Uf=0$, $Af=0$ are said to form a *complete system* of two members. When, in (9), $\lambda=0$, the two equations are said to form a *Jacobian system* of two members.

For the sake of symmetry we shall assume the two linear partial differential equations of the first order in the forms

$$A_1 f=0, \ A_2 f=0,$$

and shall merely assume that they fulfil a relation of which (9) is a particular case, that is, we shall assume that the relation

$$(A_1, A_2) \equiv \rho_1(x, y, z) A_1 f + \rho_2(x, y, z) A_2 f \text{ ......}(11)$$

exists.

If $\rho_1 \equiv 0$, the condition (11) is identical with (9).

As far as the maintenance of the condition (11) is concerned, we shall see that a condition of the form (11) must still hold when the equations $A_1 f=0$, $A_2 f=0$ are replaced by any equations which are consequences of these two, as

$$\overline{A}_1 f \equiv \lambda_1 . Af + \lambda_2 . A_2 f = 0, \ \overline{A}_2 f \equiv \mu_1 . A_1 f + \mu_2 . A_2 f = 0, \ (12)$$

where λ_i, $\mu\kappa$ are arbitrary multipliers, whose determinant, however,

$$\Delta \equiv \lambda_1 \mu_2 - \lambda_2 \mu_1$$

must evidently not be zero.

Hence

$$\begin{aligned}(\overline{A}_1f, \overline{A}_2f) &\equiv (\lambda_1A_1+\lambda_2A_2,\ \mu_1A_1+\mu_2A_2)\\ &\equiv \lambda_1\mu_1(A_1, A_1)+\lambda_1\mu_2(A_1, A_2)+\lambda_2\mu_1(A_2, A_1)+\lambda_2\mu_2(A_2, A_2)\\ &+(\lambda_1 . A_1\mu_1+\lambda_2 . A_2\mu_1)A_1f+(\lambda_1 . A_1\mu_2+\lambda_2 . A_2\mu_2)A_2f\\ &-(\mu_1 . A_1\lambda_1+\mu_2 . A_2\lambda_1)A_1f-(\mu_1 . A_1\lambda_2+\mu_2 . A_2\lambda_2)A_2f.\end{aligned}$$

Since we know that $(A_1, A_1)\equiv(A_2, A_2)\equiv 0$; and since (as is easily verified) $(A_2, A_1)\equiv-(A_1, A_2)$, while the four last terms are affected by coefficients which are functions of x, y, z, it is clear that $(\overline{A}_1f, \overline{A}_2f)$ is an expression which is linear in terms of A_1f and A_2f, that is, by means of (12), $(\overline{A}_1f, \overline{A}_2f)$ is linear in terms of $\overline{A}_1f$ and $\overline{A}_2f$.

Thus, as far as the relation (11) is concerned, it is certain that the equations $A_1f=0$, $A_2f=0$ may be replaced by any equations of the form $\overline{A}_1f=0$, $\overline{A}_2f=0$, as given in (12).

Let us therefore take the two linear partial equations in the forms,

$$\overline{A}_1f\equiv\frac{\partial f}{\partial x}-\sigma_1(x, y, z)\frac{\partial f}{\partial z}=0,\ \overline{A}_2f\equiv\frac{\partial f}{\partial y}-\sigma_2(x, y, z)\frac{\partial f}{\partial z}=0. \quad (13)$$

Here $(\overline{A}_1f_1, \overline{A}_2f)$ must still be capable of being written as a linear expression in terms of $\overline{A}_1f$ and $\overline{A}_2f$; but when the operation indicated by $(\overline{A}_1, \overline{A}_2)$ is carried out, it will be found by (13) that the result is free of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$: that is, in the expression

$$(\overline{A}_1, \overline{A}_2)\equiv\tau_1 . \overline{A}_1f+\tau_2 . \overline{A}_2f,$$

when A_1f and A_2f are chosen in the form (13), we must have $\tau_1\equiv\tau_2\equiv0$: or,

$$(\overline{A}_1f,\ \overline{A}_2f)\equiv0. \ldots\ldots\ldots\ldots\ldots\ldots\ldots(14)$$

Hence, if two linear partial differential equations of the first order satisfy a condition of the form (11), they may always be chosen in such a manner as to satisfy the condition (14).

159. It now remains for us to show that if two given linear partial differential equations of the first order satisfy a condition of the form (14), they must have a common solution.

If u and v be the solutions of $\overline{A}_1 f=0$, it is known that the most general solution of $\overline{A}_1 f=0$ must be some function of u and v of the form $\Omega(u, v)$. We now wish to determine Ω in such manner that it shall also be a solution of $\overline{A}_2 f=0$. We have

$$\overline{A}_2(\Omega(u, v)) \equiv \frac{\partial\Omega}{\partial u} \cdot \overline{A}_2(u) + \frac{\partial\Omega}{\partial v} \cdot \overline{A}_2(v);$$

and by means of the relation (14), putting u and v respectively for f in

$$\overline{A}_1(\overline{A}_2 f) - \overline{A}_2(\overline{A}_1 f) \equiv 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots(14)$$

it is easy to see that, since $\overline{A}_1(u) \equiv \overline{A}_1(v) \equiv 0$,

$$\overline{A}_1(\overline{A}_2(u)) \equiv \overline{A}_1(\overline{A}_2(v)) \equiv 0.$$

That is to say, $\overline{A}_2(u)$ and $\overline{A}_2(v)$ are solutions of $\overline{A}_1 f=0$, and are therefore functions of u and v, say,

$$\overline{A}_2(u) \equiv \phi(u, v), \quad \overline{A}_2(v) \equiv \psi(u, v).$$

Hence

$$\overline{A}_2(\Omega(u, v)) \equiv \phi(u, v)\frac{\partial\Omega}{\partial u} + \psi(u, v)\frac{\partial\Omega}{\partial v}.$$

The condition, therefore, that $\Omega(u, v)$, which is a solution of $\overline{A}_1 f=0$, shall also be a solution of $\overline{A}_2 f=0$, takes the form

$$\phi(u, v)\frac{\partial\Omega}{\partial u} + \psi(u, v)\frac{\partial\Omega}{\partial v} = 0. \quad \ldots\ldots\ldots\ldots\ldots(15)$$

This is a linear partial differential equation of the first order in u and v; and it is always satisfied by the integral function of the corresponding system

$$\frac{du}{\phi(u, v)} = \frac{dv}{\psi(u, v)}.$$

If this integral function be $W(u, v)$, then W is the common solution of $\overline{A}_1 f=0$ and $\overline{A}_2 f=0$, the existence of which was to be proved. Of course W is a function of x, y, z; so that

$$W(x, y, z)=\text{const.}$$

represents a family of surfaces in space.

160. We shall now return to our original equations of Art. 158,

$$Uf\equiv\xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y}+\zeta\frac{\partial f}{\partial z}=0,\ Af\equiv X\frac{\partial f}{\partial x}+Y\frac{\partial f}{\partial y}+Z\frac{\partial f}{\partial z}=0,\ (16)$$

having proved that the existence of the condition (9), that the equation $Af=0$ shall admit of the G_1 Uf, means that the equations (16) form a complete system—that is, that they have one solution in common.

If W be the common solution of (16), at any point x, y, z on one of the surfaces $W=$const., *two* tangential directions are assigned to the point by means of Uf and Af; and the direction cosines of these tangential directions are proportional respectively to ξ, η, ζ and X, Y, Z, Art. 19.

If α, β, γ be three quantities proportional to the direction cosines of a line perpendicular to the above two tangential directions at the point x, y, z, we have

$$X\alpha+Y\beta+Z\gamma=0,$$
$$\xi\alpha+\eta\beta+\zeta\gamma=0;$$

whence,

$$\alpha=Y\zeta-\eta Z,\quad \beta=Z\xi-\zeta X,\quad \gamma=X\eta-\xi Y.$$

If now dx, dy, dz represent the differential coefficients of the variables x, y, z on the surface $W=$const., it follows that the relation

$$(Y\zeta-\eta Z)dx+(Z\xi-\zeta X)d\eta+(X\eta-\xi Y)dz=0\ \ldots(17)$$

must be satisfied by the coordinates of all the points on those surfaces.

In other words, *the common integral surfaces of the complete system $Af=0$, $Uf=0$ satisfy the total differential equation* (17).

A method for integrating equations of the form (17) has been given in Art. 100, Chapter VIII. If

$$W(x, y, z)=\text{const.}$$

be the integral required, we know that W will be one of the solutions of the given invariant linear partial differential equation.

161. Considering one of the solutions of the linear partial differential equation of the first order in three variables which admits of a known G_1 as having been obtained, we shall now show that *the other solution may be found by a mere quadrature.*

To this end let us suppose that $W(x, y, z)$, the solution already found, actually contains the variable z—for, of course, it must contain one at least of the three variables—and in place of x, y, z, let x, y, W be introduced as new variables. In these variables Af, by Art. 35, will have the form

$$\overline{A}f \equiv Ax \,.\, \frac{\partial f}{\partial x}+Ay \,.\, \frac{\partial f}{\partial y}+A\,W \,.\, \frac{\partial f}{\partial W}:$$

or, since by hypothesis, $A(W)\equiv 0$,

$$\overline{A}f \equiv Ax \,.\, \frac{\partial f}{\partial x}+Ay \,.\, \frac{\partial f}{\partial y}.$$

Now eliminate z from Ax and Ay, and $\overline{A}f$ will have the form

$$\overline{A}f \equiv \alpha(x, y, W)\frac{\partial f}{\partial x}+\beta(x, y, W)\frac{\partial f}{\partial y}.$$

Analogously, we find for the transformed Uf,

$$\overline{U}f \equiv \gamma(x, y, W)\frac{\partial f}{\partial x}+\delta(x, y, W)\frac{\partial f}{\partial z}.$$

We see that in $\overline{A}f=0$ no differential coefficient with respect to W occurs at all; and $\overline{U}f$ does not transform this variable; hence, W plays the rôle of a mere constant, and x and y are the only variables.

The problem has now been reduced to the integration of the linear partial differential equation in two variables $\overline{A}f=0$, which admits of the known G_1, in the same two variables, $\overline{U}f$. But as this partial differential equation is equivalent to the ordinary differential equation, Art. 16,

$$\alpha dy-\beta dx=0,$$

which admits of $\overline{U}f$, the solution can be found, by the methods of Chapter IV., by a mere quadrature, in the form

$$V\equiv\int\frac{\alpha(x,\,y,\,W)dy-\beta(x,\,y,\,W)dx}{\alpha(x,\,y,\,W)\,.\,\delta(x,y,\,W)-\beta(x,y,\,W)\,.\,\gamma(x,y,\,W)}.$$

The integration here is to be performed as if W were a constant, and afterward the value of W as a function of x, y, z is to be introduced.

We may sum up the results of this section as follows: *If a linear partial differential equation of the first order in three variables admits of a known infinitesimal transformation which is not trivial, its integration may be accomplished by the integration of an ordinary differential equation of the first order in two variables, together with one quadrature.*

Example. The linear partial differential equation of the first order

$$Af\equiv(x^2+y^2+yz)\frac{\partial f}{\partial x}+(x^2+y^2-xz)\frac{\partial f}{\partial y}+(xz+yz)\frac{\partial f}{\partial z}=0$$

admits of the G_1

$$Uf\equiv x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z},$$

since the application of the criterion (9) gives in this case

$$(Uf,\,Af)\equiv Af.$$

Thus $Uf=0$ and $Af=0$ form a complete system with a solution which is the integral of the ordinary differential equation (17). The latter equation will be found to reduce itself in this case to the form

$$xzdx+yzdy-(x^2+y^2)dz=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots(21)$$

when the substitutions $X\equiv x^2+y^2+yz$, $\xi=x$, etc., are made.

By the method of Chapter VIII. we find at once as the integral of (21),

$$W(x, y, z)\equiv\frac{\sqrt{x^2+y^2}}{z}=\text{const.};$$

and it may be readily verified that W is really a solution of both $Uf=0$ and $Af=0$, since it is found that

$$U(W)\equiv A(W)\equiv 0.$$

Now we shall introduce x, y, and W as new variables, eliminating z by means of

$$z=\frac{\sqrt{x^2+y^2}}{W}.$$

Hence

$$\overline{A}f\equiv\left(x^2+y^2+\frac{y}{W}\sqrt{x^2+y^2}\right)\frac{\partial f}{\partial x}+\left(x^2+y^2-\frac{x}{W}\sqrt{x^2+y^2}\right)\frac{\partial f}{\partial y}=0\,;$$

and

$$\overline{U}f\equiv x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}.$$

Hence, the second solution of $Af=0$ is

$$V\equiv\int\frac{\left(x^2+y^2+\frac{y}{W}\sqrt{x^2+y^2}\right)dy-\left(x^2+y^2-\frac{x}{W}\sqrt{x^2+y^2}\right)dx}{\left(x^2+y^2+\frac{y}{W}\sqrt{x^2+y^2}\right)y\;-\left(x^2+y^2-\frac{x}{W}\sqrt{x^2+y^2}\right)x}$$

$$\equiv\log\left\{\sqrt{x^2+y^2}-W.(x-y)\right\}.$$

If, now, in place of W, its value—in terms of x, y, z—be put, there results finally for the second solution,

$$V\equiv\log\frac{\sqrt{x^2+y^2}(y+z-x)}{z}.$$

162. A theory of integration of an invariant linear partial differential equation in n variables—that is, of an invariant simultaneous system in n variables—analogous to the theory of this paragraph for the invariant linear partial equation in three variables, might now be developed.

But a discussion, both of that theory and of the method of integration to be employed when a linear partial equation is invariant under *more* than one known G_1, must be reserved for a later occasion.

SECTION III.

Second Method for Ordinary Differential Equations of the Second Order in Two Variables.

163. The theory of integration of the last section may be readily applied to ordinary differential equations of the second order in two variables which admit of a known G_1.

For, Art. 152, it was seen that the differential equation of the second order

$$y''-\omega(x, y, y')=0 \quad \text{......................(1)}$$

is equivalent to the simultaneous system in three variables,

$$\frac{dx}{1}=\frac{dy}{y'}=\frac{dy'}{\omega(x, y, y')}, \quad \text{..................(2)}$$

which, in turn, is equivalent to the linear partial differential equation of the first order in three variables,

$$Af\equiv\frac{\partial f}{\partial x}+y'\frac{\partial f}{\partial y}+\omega(x, y, y')\frac{\partial f}{\partial y'}=0. \quad \text{..........(3)}$$

Thus, if the differential equation (1) admits of a known twice-extended G_1 $U''f$, the partial differential equation (3) must admit of the once-extended G_1,

$$U'f\equiv\left\{\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y}+\eta'\frac{\partial f}{\partial y'}\right\},$$

that is, we must have

$$(U', A)\equiv\lambda\,.\,Af. \quad \text{.....................(4)}$$

Thus the condition (9) of the foregoing section is satisfied, and the partial differential equation (3) may be integrated by the methods of that section. That is to say, Art. 100, *if an ordinary differential equation of the second order in two variables,*

$$y''-\omega(x, y, y')=0,$$

admits of a known G_1, the differential equation of the second order may be completely integrated by the integration of an ordinary differential equation of the first order in two variables, and a quadrature.

Example. Given

$$xyy''+xy'^2-yy'=0.$$

This equation may be written

$$y''-\left(\frac{y'}{x}-\frac{y'^2}{y}\right)=0\,;\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

and hence (3) has the form

$$Af\equiv\frac{\partial f}{\partial x}+y'\frac{\partial f}{\partial y}+\left(\frac{y'}{x}-\frac{y'^2}{y}\right)\frac{\partial f}{\partial y'}=0.$$

It may be at once verified that (5) admits of the G_1

$$Uf\equiv x\frac{\partial f}{\partial x},$$

to which corresponds the once-extended G_1

$$U'f\equiv x\frac{\partial f}{\partial x}-y'\frac{\partial f}{\partial y'},$$

so that the condition (9), Sec. 2, is satisfied, and the equations

$$Af=0,\quad U'f=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

form a complete system.

The common solution of the equations (6) must be the integral-function of the total equation corresponding to (17), Sec. 2,

$$yy'\,dx-(2y-xy')dy+xy\,dy'=0.\ \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

By Art. 99, or Art. 100, the integral-function of (7) is found to be

$$W(x, y, y')\equiv y^2-xyy'.\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Now introduce x, y, and W—the common solution of the equations (6)—as new variables; thus,

$$\overline{A}f \equiv Ax\frac{\partial f}{\partial x} + Ay\frac{\partial f}{\partial y}, \quad \overline{U}f \equiv Ux\frac{\partial f}{\partial x} + Uy\frac{\partial f}{\partial y};$$

or

$$\overline{A}f \equiv \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y}, \qquad \overline{U}f \equiv x\frac{\partial f}{\partial y}.$$

Eliminating y' from $\overline{A}f$ by means of (8), we find

$$\overline{A}f \equiv \frac{\partial f}{\partial x} + \frac{y^2 - W}{xy}\,\frac{\partial f}{\partial x}.$$

Now $\overline{A}f = 0$, a linear partial equation in the variables x and y, admits of $\overline{U}f$ in the same variables: hence the second solution of $Af = 0$ is found by a quadrature in the form

$$V \equiv \frac{x^2}{y^2 - W} \equiv \frac{x}{yy'}.$$

Thus, eliminating y' between

$$W \equiv y^2 - xyy' = c_1 \text{ and } V \equiv \frac{x}{yy'} = c_2;$$

we find

$$y^2 - \frac{x^2}{c_2} = c_1,$$

or as it may be written

$$mx^2 + ny^2 = 1, \qquad (m,\ n \text{ consts.})$$

which is the form of the complete integral of (5).

Thus we see that (5) represents the ∞^2 conic sections whose axes coincide with the axes of coordinates; and it is clear, geometrically, that this family of ∞^2 curves is invariant under the G_1 of affine transformations

$$Uf \equiv x\frac{\partial f}{\partial x}.$$

164. The simultaneous systems given in the following Examples are simple; and, for the most part, they may be integrated by the methods of both Sec. I. and Sec. II. Examples (1) to (4) are, however, intended to illustrate Arts. 148, 149; Examples (5) to (14), Arts. 150-153; while the remaining Examples are intended to be treated by the method of Sec. II., after it has been verified in each case, Art. 156, that the given simultaneous system is invariant under the accompanying G_1. Examples illustrating Sec. III. may be found at the end of Chapter IX.

EXAMPLES.

(1) $\dfrac{dx}{x^2-y^2-z^2}=\dfrac{dy}{2xy}=\dfrac{dz}{2xz}.$

(2) $\dfrac{l\,dx}{mn(y-z)}=\dfrac{m\,dy}{nl(z-x)}=\dfrac{n\,dz}{lm(x-y)}.$

(3) $\dfrac{l\,dx}{(m-n)yz}=\dfrac{m\,dy}{(n-l)zx}=\dfrac{n\,dz}{(l-m)xy}.$

(4) $\dfrac{dx}{x(y-z)}=\dfrac{dy}{y(z-x)}=\dfrac{dz}{z(x-y)}.$

(5) $\dfrac{dx}{-1}=\dfrac{dy}{3y+4z}=\dfrac{dz}{2y+5z}.$

(6) $\dfrac{dx}{z+2y-2x}=\dfrac{dy}{z^2-x-5y}=\dfrac{dz}{z}.$

(7) $\dfrac{dx}{16x+y}=\dfrac{dy}{4(3y-x)}=\dfrac{dt}{-4}.$

(8) $\dfrac{dx}{dt}+5x+y=e^t\,;\ \dfrac{dy}{dt}+3y-x=e^{2t}.$

(9) $\dfrac{dx}{2y-5x+e^t}=\dfrac{dy}{x-6y+e^{2t}}=dt.$

(10) $4\dfrac{dx}{dt}+9\dfrac{dy}{dt}+2x+31y=e^t,\quad 3\dfrac{dx}{dt}+7\dfrac{dy}{dt}+x+24y=3.$

(11) $4\dfrac{dx}{dt}+9\dfrac{dy}{dt}+44x+49y=t,\quad 3\dfrac{dx}{dt}+7\dfrac{dy}{dt}+34x+38y=e^t.$

(12) $\dfrac{d^2x}{dt^2}+n^2x=0,\quad \dfrac{d^2y}{dt^2}-n^2x=0.$

(13) $\dfrac{d^2x}{dt^2}-3x-4y+3=0,\quad \dfrac{d^2y}{dt^2}+x-8y+5=0.$

(14) $2\dfrac{d^2y}{dx^2}-\dfrac{dz}{dx}-4y=2x,\quad 2\dfrac{dy}{dx}+4\dfrac{dz}{dx}-3z=0.$

(15) $\dfrac{dx}{x-y-z+2}=\dfrac{dy}{2(y-x+z)}=\dfrac{dz}{x-y-z}\,;\quad Uf\equiv\dfrac{\partial f}{\partial x}+\dfrac{\partial f}{\partial z}.$

(16) $\dfrac{dx}{xz+e^{y-xz}}=\dfrac{dy}{-z(1+x)}=\dfrac{dz}{z(e^{y-xz}-z)}$; $Uf\equiv\dfrac{x}{1+x}\dfrac{\partial f}{\partial x}-\dfrac{z}{1+x}\dfrac{\partial f}{\partial z}.$

(17) $\dfrac{dx}{x+y}=\dfrac{dy}{x+y}=\dfrac{dz}{-(x+y+2z)}$; $Uf\equiv(x+y)\dfrac{\partial f}{\partial x}+(x+y)\dfrac{\partial f}{\partial y}+2z\dfrac{\partial f}{\partial z}.$

(18) $\dfrac{dx}{xz-y}=\dfrac{dy}{yz-x}=\dfrac{dz}{1-z^2}$; $Uf\equiv x\dfrac{\partial f}{\partial x}+y\dfrac{\partial f}{\partial y}.$

(19) $\dfrac{dx}{z-2x}=\dfrac{dy}{xz+yz+2x-z}=\dfrac{dz}{z}$; $Uf\equiv e^z\dfrac{\partial f}{\partial y}.$

(20) $\dfrac{dx}{y-z}=\dfrac{dy}{y-z}=\dfrac{dz}{(x-y)(x-z)}$; $Uf\equiv\dfrac{\partial f}{\partial x}+\dfrac{\partial f}{\partial y}+\dfrac{\partial f}{\partial z}.$

(21) Verify that the linear partial differential equation corresponding to the simultaneous system,

$$\frac{dx}{a_1x+b_1y+c_1z+d_1}=\frac{dy}{a_2x+b_2y+c_2z+d_2}=\frac{dz}{a_3x+b_3y+c_3z+d_3},$$

admits of a G_1 of the form,

$$Uf\equiv(x+a)\frac{\partial f}{\partial x}+(y+\beta)\frac{\partial f}{\partial y}+(z+\gamma)\frac{\partial f}{\partial z},$$

where a, β, γ are certain constants, and that therefore the above simultaneous system may usually be integrated by the method of Sec. II.

(22) The method of Sec. II. fails for the preceding example only in the case of Uf being *trivial*. For what values of the constants $a_1, b_1, \ldots, d_3$ is Uf trivial? What are then the integrals of the simultaneous system?

(23) Verify that the linear partial differential equation corresponding to the simultaneous system,

$$\frac{dx}{X}=\frac{dy}{Y}=\frac{dz}{Z},$$

admits of the G_1

$$Uf\equiv x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z},$$

if X, Y, Z are *homogeneous* functions of x, y, z; and that therefore, when Uf is not trivial, the integration of the above system may be reduced by the method of Sec. II.

(24) Verify that the linear partial differential equations corresponding to the simultaneous systems in Ex. (16) and (18) admit, respectively, of the G_1's,

$$Uf\equiv\frac{\dfrac{\partial f}{\partial x}+z(1+x)\dfrac{\partial f}{\partial y}+z\dfrac{\partial f}{\partial z}}{z(1+x)};$$

$$Uf\equiv(x^2+y^2)\frac{\partial f}{\partial x}+2xy\frac{\partial f}{\partial y}-y(1-z^2)\frac{\partial f}{\partial z}.$$

(25) Verify that the linear partial differential equation,

$$Af\equiv\frac{\partial f}{\partial x}+z\frac{\partial f}{\partial y}+\left(\frac{y-xz}{x}\right)^3\phi\left(\frac{y}{x}\right)\frac{\partial f}{\partial z}=0,$$

admits of the G_1,

$$Uf\equiv x^2\frac{\partial f}{\partial x}+xy\frac{\partial f}{\partial y}+(y-xz)\frac{\partial f}{\partial z}.$$

ANSWERS.

CHAPTER I.

(1) $y' = \frac{y}{x}$.

(2) $y = xy' + \sqrt{1+y'^2}$.

(3) $y' = \frac{1+y^2}{xy(1+x^2)}$.

(4) $yy'^2 + 2xy' = y$.

(5) $(1+x^2)y' + y = \tan^{-1}x$.

(6) $xy' + y = y^2 \log x$.

(7) $y = xy' + y' - y'^3$.

(8) $x^2y'^2 = 1 + y'^2$.

(9) $x^2y'' - 2xy' + 2y = 0$.

(10) $y'' + m^2y = 0$.

(11) $x^3y'' + (y - xy')^2 = 0$.

(12) $x^2y'' - xy' = 3y$.

(13) $y''' = 7y' - 6y$.

(14) $y''' - 2y'' + y' = e^x$.

(15) $y^2(1+y'^2) = r^2$.

(16) $r^2y''^2 = (1+y'^2)^3$.

(17) $1 + y'^2 - (y - xy')^2 = 0$.

(18) $xyy'' + xy'^2 - yy' = 0$.

(19) $x^2y'' - xy' + y = 0$.

CHAPTER II.

(1) $mx^2y + cy + 2 = 0$.

(2) $(1+x)(1+y) = c$.

(3) $\log\frac{x}{y} - \frac{y+x}{xy} = c$.

(4) $3(x^2 - y^2) + 2(x^3 - y^3) = c$.

(5) $\cos y = c \cos x$.

(6) $\log\left[(y + \sqrt{1+y^2})\sqrt{1+y^2}\right] = \frac{x}{\sqrt{1+x^2}} + c$.

(7) $\sin^2 x + \sin^2 y = c$.

(8) $y = c_1 x$; $yz = c_2$.

(9) $x^2 + y^2 = {c_1}^2$; $\tan^{-1}\frac{y}{x} - \log z = c_2$.

(10) $x^2+y^2=c_1^{\,2}$; $\tan{}^{1}\frac{y}{x}-\tan^{-1}z=$ const., or taking the tangent of both sides, $\frac{xz-y}{x+yz}=c_2$.

(11) $x^2-y^2=c_1$; $y^2-z^2=c_2$. (12) $x=c_1$; $y^2+z^2=c_2$.

CHAPTER III.

N.B.—Only such invariant points and lines as are within a finite distance from the origin will be taken into consideration.

(1) No invariant point. An Invariant is $\Omega(y)$.

(2) An Invariant is $\Omega(x)$.

(3) All points on the y-axis invariant. An Invariant is $\Omega(y)$.

(4) The origin is an invariant point. An Invariant is $\Omega\left(\frac{y}{x}\right)$.

(5) The origin is an invariant point. An Invariant is $\Omega\left(\frac{y}{x}\right)$.

(6) The origin is an invariant point. An Invariant is $\Omega(xy)$.

(7) The origin is an invariant point. An Invariant is $\Omega(x^2+y^2)$.

(8) All points on the y-axis are invariant. An Invariant is $\Omega\left(\frac{y}{x}\right)$.

(9) All points on the x-axis are invariant. An Invariant is $\Omega\left(\frac{y}{x}\right)$.

CHAPTER IV.

(1) $x^3-6x^2y-6xy^2+y^3=c$. (2) $x^2-y^2=c\,.\,y^3$.

(3) $x+ye^{\frac{x}{y}}=c$. (4) $\cos(mx+ny)+\sin(nx+my)=c$.

(5) $\sqrt{1+x^2+y^2}+\tan^{-1}\frac{x}{y}=c$. (6) $e^x(x^2+y^2)=c$.

(7) $y=c\,.\,e^{-\frac{x}{y}}$. (8) $y=ce^{-\sqrt{\frac{x}{y}}}$.

(9) $x^2=c^2+2cy$. (10) $\log(x^2+y^2)=2\tan^{-1}\frac{x}{y}+c$.

(11) $x=ce^{-\sin\frac{y}{x}}$. (12) $(y+x)^2(y+2x)^3=c$.

(13) $\sin^{-1}\frac{y}{x}=\log x+c$. (14) $x^2y^4=y^2x^4+c$.

(15) $x^3+y^3=cxy$. (16) $x^2-xy+y^2+x-y=c$.

(17) $(y-x+1)^2(y+x-1)^5=c.$

(18) $\frac{x^2+y^2}{2}+\tan^{-1}\frac{y}{x}=c.$

(19) $xy^2=c(x+2y).$

(20) $y=cx.$

(21) $y=ce^{xy}.$

(22) $xy-\frac{1}{xy}=\log cy^2.$

(23) $xy=c.$

(24) $xy+\log\sin(xy)=\log\frac{cx}{y}.$

(25) $y=cx^a+\frac{x}{1-a}-\frac{1}{a}.$

(26) $y=\frac{x}{\sqrt{1-x^2}}+ce^{\frac{-x}{\sqrt{1-x^2}}}.$

(27) $x=\tan^{-1}y-1+c\,.\,e^{-\tan^{-1}y}.$

(28) $\frac{1}{y^2}=x+\frac{1}{2}+ce^{2x}.$

(29) $\frac{1}{y}=c\sqrt{1-x^2}-1.$

(30) $\frac{1}{y}=\log x+1+cx.$

(31) Admits of $Uf\equiv 2x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}.$ Ans. $\log x+\frac{y^2}{x}=c.$

(32) $y\sqrt{1+x^2}=\log\frac{\sqrt{1+x^2}-1}{x}+c.$

(33) $y(\sec x+\tan x)=x+c.$

(34) $y=\{c\sqrt{1-x^2}-a\}^{-1}.$

(35) $y=\{ce^{2x^2}+x^2+\frac{1}{2}\}^{-\frac{1}{2}}.$

(36) $y=\{cx+\log x+1\}^{-1}.$

(37) $y=\frac{\tan x+\sec x}{\sin x+c}.$

(38) $5y^2=2\sin x+4\cos x+ce^{-2x}.$

(39) Admits of $Uf\equiv\frac{x}{m}\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}.$ Ans. $\frac{y}{x}-\frac{x^{m-1}}{m-1}=c.$

(40) $y^\kappa=c\,.\,x.$

(41) $y^a=c\,.\,e^x.$

(42) $y^2=4ax+c$ (parabola).

(43) $\rho=c\,.\,(1-\cos\theta).$

(44) $y^2=\kappa\,.\,x^{n+1}+c.$

(45) $x^2+y^2=2cx.$

(46) $y^\kappa=ce^x.$

(47) $(x-y)^2-2cy=0$ (parabola).

(48) $x^2+y^2=\kappa(x-c)^2$ (a conic).

(49) $y=\frac{x}{2}\left\{\left(\frac{c}{x}\right)^\kappa-\left(\frac{x}{c}\right)^\kappa\right\}.$

(50) $c\,.\,y^m=(m-1)x+ny.$

(51) $\frac{xy'-y}{y'}=F(y).$

(52) $xy'-y=F(x).$

CHAPTER V.

(1) $x^2+y^2=c^2.$

(2) $2x^2+y^2=c^2.$

(3) $y=cx.$

(4) The system is self-orthogonal.

(5) $x^2+y^2=2a^2\log x+c$.

(6) The differential equation is homogeneous; hence the orthogonal trajectories are,

$$\log(x-y)-\int\frac{d\left(\frac{y}{x}\right)}{\left(\frac{y}{x}-1\right)\left(1+\frac{y}{x}+\sqrt{2\frac{y}{x}}\right)}=\text{const.}$$

(7) $y^{b^2}=c\,.\,x^{a^2}$.

(8) Isothermal. $x^2+y^2+cy+1=0$.

(9) The system is self-orthogonal.

(10) $y=\int\frac{dx}{\phi'(x)}+\text{const.}$ (11) $y=-2\sqrt{nx}+c$.

(13) $\frac{2}{\rho}=\sin^2\theta+c$. (14) $\theta=-\int\frac{d\rho}{\rho^2\phi'(\rho)}+\text{const.}$

(15) $\theta=\left\{\cos^{-1}\frac{\rho}{b}-\frac{\sqrt{b^2-\rho^2}}{\rho}\right\}+\text{const.}$

CHAPTER VI.

(1) $(y-2x+c)(y-3x+c)=0$. (2) $y=c\,.\,e^{ax}$, $y=c\,.\,e^{-ax}$.

(3) $(xy+c)(x^2y+c)=0$. (4) $(x^2-2y+c)\{(x+y-1)e^x+c\}=0$.

(5) $(y+c)(y+x^2+c)(xy+cy+1)=0$.

(6) $y^2\sin^2x+2cy+c^2=0$.

(7) Admits of $Uf\equiv(1+x)\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}$.
General integral, $c^2+2c(1+x+y)+(1+x-y)^2=0$.
Singular solution, $y=0$.

(8) $y^2=2cx+c^2$. Singular solution, $x^2+y^2=0$.

(9) $x^2+c(x-3y)+c^2=0$. Singular solution, $(x+3y)(x-y)=0$.

(10) $x^2+y^2-4cx+3c^2=0$. Singular solution, $x^2-3y^2=0$.

(11) $y=\frac{c}{x}+c^2$. Singular solution, $1+4x^2y=0$.
Admits of $Uf\equiv x\frac{\partial f}{\partial x}-2y\frac{\partial f}{\partial y}$.

(12) $2y=cx^2+\frac{a}{c}$. Singular solution, $y^2=ax^2$.

(13) $x\pm\sqrt{a^2+4by}=a\log(a\pm\sqrt{a^2+4by})+c$. Singular solution, $y=0$.

(14) Admits of $Uf \equiv x\frac{\partial f}{\partial x}+2y\frac{\partial f}{\partial y}$.

General integral,

$$c(2y \pm x\sqrt{x^2+y})^{\sqrt{17}}=\frac{\pm 4\sqrt{x^2+y}-x(\sqrt{17}-1)}{\pm 4\sqrt{x^2+y}+x(\sqrt{17}+1)}.$$

Singular solution, $x^4+x^2y-4y^2=0$.

(15) $\left\{y\sqrt{y^2-x^2}-x^2\log\frac{y+\sqrt{y^2-x^2}}{x}\right\}^2=(y^2+x^2\log cx^2)^2$.

Singular solution, $x=0$.

(16) $4(x+c)^3+(x+c)^2-18y(x+c)-27y^2-4y=0$.

Singular solution, $y=0$.

(17) $y^2=2cx+c^2$. Singular solution, $x^2+y^2=0$.

(18) $x+1=\pm\sqrt{2y+c}+\log(\pm\sqrt{2y+c}-1)$.

(19) $e^{2y}+2cxe^y+c^2=0$. Singular solution, $x=\pm 1$.

(20) $c\{nx^2+2y^2 \pm x\sqrt{n^2x^2+4my^2}\}^n=\{(2m-n)x \pm \sqrt{n^2x^2+4my^2}\}^{2m}$.

(21) $(x^2-y^2+c)(x^2-y^2+cx^4)=0$.

(22) $(y-x-\mathrm{c})(x^2+y^2-c)=0$.

(23) $x^2=c(y-c)$. Singular solution, $y=\pm 2x$.

(24) Admits of $Uf \equiv x\frac{\partial f}{\partial x}+4y\frac{\partial f}{\partial y}$. General integral, $y=c^2(x-c)^2$.

Singular solution, $x^4-16y=0$.

(25) $(y+c)^2=x(x-a)(x-b)$. Singular solution, $x(x-a)(x-b)=0$.

(26) $c^2+2cx(3a^2y^2-8x^2)-3x^2a^4y^4+a^6y^6=0$. No singular solution.

(27) Admits of $Uf \equiv x\frac{\partial f}{\partial x}+4y\frac{\partial f}{\partial y}$. Singular solution, $y=\frac{-x^4}{4}$.

CHAPTER VII.

(1) The answer is given by (11) in connection with (12), Art. 91, since $\lambda=1$.

(2) The answer is given by (11) in connection with (13), since $\lambda=2$.

(3) Use (14) and (16). General integral is:

$$\log\frac{3yx^{\frac{1}{3}}-3x^{\frac{2}{3}}-y}{3yx^{\frac{1}{3}}+3x^{\frac{2}{3}}+y}+6x^{\frac{1}{3}}=c.$$

(4) and (5). See Art. 92.

(6) $y=cx+\frac{m}{c}$. Singular solution, $y^2=mx$.

(7) $y=cx+\sqrt{b^2+a^2c^2}$. Singular solution, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

(8) $y=cx+c-c^3$. Singular solution, $27y^2=4(1+x)^3$.

(9) $(y-cx)^2=1+c^2$. Singular solution, $x^2+y^2=1$.

(10) $y=c(x-1)-c^2$. Singular solution, $4y=(x-1)^2$.

(11) $(y-cx)^2+4c=0$. Singular solution, $xy=1$.

(12) $(y-cx)(ac-b)=abc$. Singular solution, $\left(\frac{x}{a}\right)^{\frac{1}{2}}\pm\left(\frac{y}{b}\right)^{\frac{1}{2}}=1$.

(13) $x^{\frac{1}{2}}+y^{\frac{1}{2}}=a^{\frac{1}{2}}$. A parabola.

(14) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$.

(15) $xy=\frac{a^2}{2}$. Equilateral hyperbola.

(16) $x^2=4a(a-y)$. A parabola.

CHAPTER VIII.

(1) $yz+zx+xy=c$.

(2) $\frac{x^2+y^2}{z^2}=c$.

(3) $x^2+2y^2-6xy-2xz+z^2=c$.

(4) $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=c_1$.

(5) $x=\frac{z}{y+a}+c$.

(6) $y(x+z)=c(y+z)$.

(7) $e^{x^2}(x+y+z^2)=c$.

CHAPTER IX.

(1) $y^2=x^2+c_1x+c_2$.

(2) $c_1{}^2-2c_1xy+y^2-c_2(1-x^2)=0$.

(3) $y=(x-2)e^x+c_1x+c_2$.

(4) $y=\frac{x^3}{6}-\sin x+c_1+c_2x$.

(5) $ay=\sqrt{2ax-x^2}+c_1x+c_2$.

(6) For $y''=+a^2y$, $ax=\log(y+\sqrt{y^2+c_1})+c_2$;

for $y''=-a^2y$, $ax=\sin^{-1}\frac{y}{c_1}+c_2$.

(7) $(c_1x+c_2)^2+a=c_1y^2$.

(8) $3x=2a^{\frac{1}{4}}(y^{\frac{1}{2}}-2c_1)(y^{\frac{1}{2}}+c_1)^{\frac{1}{2}}+c_2$.

(9) $\frac{2y}{a}=c_1e^{\frac{x}{a}}+c_1^{-1}e^{-\frac{x}{a}}+c_2.$ (10) $c_2e^y=\cos(x+c_1).$

(11) $y=\log\sin(x-c_1)+c_2.$ (12) $(x+c_1)^2+(y+c_2)^2=a^2.$

(13) $y=c_1\log x+c_2.$ (14) $y=\frac{c_1x^2}{2}+xf(c_1)+c_2.$

(15) $y=c_1x+(c_1^2+1)\log(x-c_1)+c_2.$

(16) $y=x+c_1\{\sin^{-1}x+x\sqrt{1-x^2}\}+c_2.$

(17) $y=\frac{x^{n+1}}{(n+1)^2}+c_1\log x+c_2.$

(18) $y=c_1\sqrt{a^2-x^2}+\frac{x^2}{2a}+c_2.$ (19) $\frac{c_1+y}{c_1-y}=e^{c_1(x+c_2)}.$

(20) $\log y=1+\frac{1}{c_1x+c_2}.$ (21) $y=\frac{c_1}{x}+c_2x^2.$

(22) $y=-\log(c_2-c_1\log x).$ (23) $y=c_1x^3+\frac{c_2}{x}.$

(24) $c_1x^2+c_2y^2=1.$ (25) $x^2+y^2=c_1e^{c_2\tan^{-1}\frac{y}{x}}.$

(26) $x^2+2c_1x+y^2+2c_2y=0.$ $\left(\text{Admits of } Uf\equiv-y\frac{\partial f}{\partial x}+x\frac{\partial f}{\partial y}.\right)$

(27) $c_1y^2-\log y^2=4c_1(x+c_2).$ (28) $c_1y^2-\frac{c_1^2}{n}(x+c_2)^2=1.$

(29) $(y-c_2)^2+(x+ac_1)^2=a^2.$

(30) (*a*) $y^2+x^2=2c_1x+c_2$;
(*b*) $x=c_2+c_1\log\{y+\sqrt{y^2-c_1^2}\}$ (a catenary).

(31) (*a*) $x+c_2=c_1\operatorname{vers}^{-1}\frac{y}{c_1}-\sqrt{2c_1y-y^2}$ (a cycloid).
(*b*) $(x+c_2)^2=2c_1y-c_1^2$ (a parabola).

(32) It is geometrically evident that the family of curves admit of the translations $Uf\equiv\frac{\partial f}{\partial y}.$

(33) It is geometrically evident that the family of curves admits of the group of similitudinous transformations

$$Uf\equiv x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}.$$

CHAPTER X.

(1) $y=c_1+c_2x+c_3x^2+x^2\log x.$ (2) $y=c_1e^{\frac{x}{a}}+c_2x+c_3.$

(3) $y=c_1+c_2x+c_3x^2+c_4x^3-(x+a)^2\log\sqrt{x+a}.$

(4) $y=c_1+c_2x+c_3x^2+c_4x^3+x\cos x-4\sin x.$

(5) $y=c_1(\log x-1)x+c_2x^4+c_3x+c_4.$

(6) $y=c_1+c_2x+c_3x^2+c_4x^3+e^{-x}\cos x.$

(7) $y=c_1+c_2x+c_3x^2+\ldots\ldots+c_nx^{n-1}+(x-n)e^x.$

(8) $y=c_1+c_2x+c_3x^2+\dfrac{7\cos x}{9}-\dfrac{\cos^3 x}{27}.$

(9) $y=\dfrac{x^3}{6}-\sin x+c_1x^2+c_2x+c_3.$ (10) See Art. 132.

(11) $y=\dfrac{4}{15c_1}(x+c_1{}^2a^2)^{\frac{5}{2}}+c_2x+c_3.$

(12) $2y\sqrt{c_1}=(x+c_3)\sqrt{(x+c_3)^2+c_1{}^2}+c_1{}^2\log(x+c_3+\sqrt{(x+c_3)^2+c_1{}^2})+c_2.$

(13) $y=c_1\log x+c_2x^2+c_3x+c_4.$

(14) $12y=(x+c_3)^3+c_1(x+c_3)-6(x+c_3)\log(x+c_3)+c_2.$

CHAPTER XI.

(1) $y=c_1e^{ax}+c_2e^{-ax}.$ (2) $y=c_1e^{3x}+c_2e^{\frac{x}{3}}.$

(3) $y=c_1e^{2x}+c_2e^{-2x}+c_3.$ (4) $y=c_1e^{2x}+c_2e^{-3x}+c_3e^x.$

(5) $y=c_1e^{3x}+c_2e^{-3x}+c_3e^{x\sqrt{3}}+c_4e^{-x\sqrt{3}}.$

(6) $y=e^x(c_1+c_2x+c_3x^2+c_4x^3).$

(7) $y=e^{2abx}\{c_1\sin(a^2-b^2)x+c_2\cos(a^2-b^2)x\}.$

(8) $ye^x=c_1\,.\,e^{2x}+c_2\sin x\sqrt{2}+c_3\cos x\sqrt{2}.$

(9) $y=c_1\,.\,e^{x\sqrt{2}}+c_2e^{-x\sqrt{2}}+c_3\sin 2x+c_4\cos 2x.$

(10) $y=c_1e^{-x}+(c_2+c_3x)e^{2x}.$

(11) $y=(c_1+c_2x)\cos nx+(c_3+c_4x)\cos 2x.$

(12) $y=(c_1+c_2x+c_3x^2)e^x+c_4.$ (13) $y=c_1e^{3x}+c_2e^{4x}+\dfrac{12x+7}{144}.$

(14) $y=c_1\sin x+c_2\cos x+(c_3+c_4x)e^x+1.$

(15) $y=\left(c_1+c_2x+\dfrac{x^2}{2}\right)e^x+c_3.$

(16) $y=c_1\sin nx+c_2\cos nx+\dfrac{1+x+x^2}{n^2}-\dfrac{2}{n^4}.$

(17) $y=\left(c_1+c_2x+\frac{x^2}{2}\right)e^x.$

(18) $y=c_1e^x+c_2e^{2x}+\frac{xe^{nx}}{n^2-3n+2}-\frac{(2n-3)e^{nx}}{(n^2-3n+2)^2}.$

(19) $y=\left(c_1-\frac{x^2}{16}\right)\sin 2x+\left(c_2-\frac{x}{32}\right)\cos 2x+\frac{x}{8}.$

(20) $y=c_1e^{-2x}+\left(c_2-\frac{x}{20}\right)e^x\cos x+\left(c_3+\frac{3x}{20}\right)e^x\sin x.$

(21) $y=c_1x^3+\frac{c_2}{x}.$

(22) $y=c_1(x+a)^2+c_2(x+a)^3+\frac{3x+2a}{6}.$

(23) $y=x(c_1\sin\log x+c_2\cos\log x+\log x).$

(24) $y=(2x-1)\left\{c_1+c_2(2x-1)^{\frac{\sqrt{3}}{2}}+c_3(2x-1)^{-\frac{\sqrt{3}}{2}}\right\}.$

(25) A particular integral-function is x.

General integral :

$$y=-\frac{x}{9}(1-x^2)^{\frac{3}{2}}+c_1\{x\sin^{-1}x+(1-x^2)^{\frac{1}{2}}\}+c_2x.$$

(26) A particular integral-function is e^x.

General integral :

$$y=e^x\left\{\int e^{\frac{x^2-4x}{2}}\left(\int xe^{\frac{2x-x^2}{2}}dx+c_1\right)dx+c_2\right\}.$$

(27) A particular integral-function is $\frac{1}{x}$.

General integral : $y=\frac{c_2}{x}+\frac{c_1+e^x}{x^2}.$

(28) A particular integral-function is $\frac{1}{x}$.

General integral : $y=c_1+c_2\log x+\frac{x^2}{4}.$

(29) A particular integral-function is $e^{\sin^{-1}x}$.

General integral : $y=c_1e^{\sin^{-1}x}+c_2e^{\cos^{-1}x}.$

(30) A particular integral-function is e^x.

General integral : $y=c_1e^x+c_2x-(x^2+x+1).$

CHAPTER XII.

(1) $y=c_1z$; $x^2+y^2+z^2=c_2 . z$.

(2) $l^2x+m^2y+n^2z=c_1$; $l^2x^2+m^2y^2+n^2z^2=c_2$.

(3) $lx^2+my^2+nz^2=c_1$; $l^2x^2+m^2y^2+n^2z^2=c_2$.

(4) $x+y+z=c_1$; $xyz=c_2$.

(5) $y=-2c_1e^{-x}+c_2e^{-7x}$; $z=c_1e^{-x}+c_2e^{-7x}$.

(6) $x=c_1z^{-4}+2c_2z^{-3}+\frac{3z}{10}+\frac{z^2}{15}$; $y=-c_1z^{-4}-c_2z^{-3}-\frac{z}{20}+\frac{2z^2}{15}$.

(7) $2x=(2c_2-c_1-c_2t)e^{-\frac{7t}{2}}$; $y=(c_1+c_2t)e^{-\frac{7t}{2}}$.

(8) $x=(c_1+c_2t)e^{-4t}-\frac{e^{2t}}{36}+\frac{4e^t}{25}$; $y=-(c_1+c_2+c_2t)e^{-4t}+\frac{7e^{2t}}{36}+\frac{e^t}{25}$.

(9) $x=2c_1e^{-4t}-c_2e^{-7t}+\frac{e^{2t}}{27}+\frac{7e^t}{40}$; $y=c_1e^{-4t}+c_2e^{-7t}+\frac{7e^{2t}}{54}+\frac{e^t}{40}$.

(10) $x=(c_1\sin t+c_2\cos t)e^{-4t}+\frac{31e^t}{26}-\frac{93}{17}$;

$y=\{(c_2-c_1)\sin t-(c_2+c_1)\cos t\}e^{-4t}-\frac{2e^t}{13}+\frac{6}{17}$.

(11) $x=c_1e^{-t}+c_2e^{-6t}-\frac{29e^t}{7}+\frac{19t}{3}-\frac{56}{9}$;

$y=-c_1e^{-t}+4c_2e^{-6t}+\frac{24e^t}{7}-\frac{17t}{3}+\frac{56}{9}$.

(12) $x=c_1\sin nt+c_2\cos nt$; $y=c_3+c_4t-x$.

(13) $x=4c_1e^{2t}+4c_2e^{-2t}+c_3e^{t\sqrt{7}}+c_4e^{-t\sqrt{7}}+\frac{1}{7}$,

$y=c_1e^{2t}+c_2e^{-2t}+c_3e^{t\sqrt{7}}+c_4e^{-t\sqrt{7}}+\frac{9}{14}$.

(14) $y=(c_1+c_2x)e^x+3c_3e^{-\frac{3x}{2}}-\frac{x}{2}$; $z=2(3c_2-c_1-c_2x)e^x-c_3e^{-\frac{3x}{2}}-\frac{1}{3}$.

(15) $y+2z=c_1$; $2x+y-2\log(x-y-z+1)=c_2$.

(16) $e^{xy}+e^y=c_1$; $x+y-\log z=c_2$.

(17) $x-y=c_1$; $xz+yz+xy=c_2$. (18) $y+xz=c_1$; $x+yz=c_2$.

(19) $x=c_1z^{-2}+\frac{z}{3}$; $y=c_2e^z-c_1z^{-2}-\frac{z}{3}$.

(20) $x-y=c_1$;

$$\frac{(x-y)(x-z)}{1-x+y}+\left(\frac{x-y}{1-x+y}\right)^2\log\{(1-x+y)(x-z)-x+y\}-z=c_2$$

(22) The G_1 is trivial for

$$a_1=b_2=c_3=\frac{d_1}{\alpha}=\frac{d_2}{\beta}=\frac{d_3}{\gamma},$$

with the rest of the constants zero. The integrals in this case are

$$\frac{x+\alpha}{y+\beta}=\text{const.},\quad \frac{y+\beta}{z+\gamma}=\text{const.}$$